STRUCTURE AND PROPERTIES OF WOOD-BASED MATERIALS

13.8.07 ①

Structure
- Atomic
- Molecular
- Fibrillar
- Cellular
- Macroscopic

Properties
- Extensive S_v
- Intensive — Independent of extension, valid locally
- Specific — Material properties

Anisotropy
- Homogenity
- Isotropy — Anisotropy, Orthotropy

Periodic Variation

Co-ordinate systems
- Rectangular — Cartesian
- Cylindrical
- Spherical

13.8.07 ②

$\boxed{\text{Stiffness} \quad \frac{d\sigma}{d\varepsilon}}$

$$Q_{ijkl} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}}$$

$\boxed{\text{Stress} \quad \frac{\partial F}{\partial A}}$

$$\sigma_{ij} = \frac{\partial F_i}{\partial A_j}$$

$$dF_i = \frac{\partial F_i}{\partial A_j} dA_j = \frac{\partial F_i}{\partial A_1} dA_1 + \frac{\partial F_i}{\partial A_2} dA_2 + \frac{\partial F_i}{\partial A_3} dA_3$$

$$= \sigma_{i1} dA_1 + \sigma_{i2} dA_2 + \sigma_{i3} dA_3$$

$\boxed{\text{Strain} \quad \frac{\partial u}{\partial x}}$

$$\varepsilon_{kl} = \frac{\partial u_k}{\partial x_l} dx_l$$

$$= \dots$$

How can we determine σ_{ij} ?

$$d\sigma_{ij} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} d\varepsilon_{kl} = Q_{ijkl}\, d\varepsilon_{kl}$$

$$= Q_{ij11} d\varepsilon_{11} + Q_{ij12} d\varepsilon_{12} + Q_{ij13} d\varepsilon_{13} +$$
$$Q_{ij21} d\varepsilon_{21} + Q_{ij22} d\varepsilon_{22} + Q_{ij23} d\varepsilon_{23} +$$
$$Q_{ij31} d\varepsilon_{31} + Q_{ij32} d\varepsilon_{32} + Q_{ij33} d\varepsilon_{33}$$

STIFFNESS MATRIX

$$[Q] \equiv \begin{bmatrix} Q_{1111} & Q_{1122} & Q_{1133} & Q_{1112} & Q_{1113} & Q_{1121} & Q_{1123} & Q_{1131} & Q_{1132} \\ Q_{2211} & Q_{2222} & Q_{2233} & Q_{2212} & Q_{2213} & Q_{2221} & Q_{2223} & Q_{2231} & Q_{2232} \\ Q_{3311} & Q_{3322} & Q_{3333} & Q_{3312} \ldots & & & & & \\ Q_{1211} & \ldots & & & & & & & \\ Q_{1311} & \ldots & & & & & & & \\ Q_{2111} & \ldots & & & & & & & \\ Q_{2311} & \ldots & & & & & & & \\ Q_{3111} & \ldots & & & & & & & \\ Q_{3211} & \ldots & & & & & & & \end{bmatrix}$$

Total of $9 \times 9 = 81$ components

Stress vector, Stiffness matrix, Strain vector

$$\bar{\sigma} = [Q] \cdot \bar{\varepsilon}$$

In component form

$$\sigma_{ij} = Q_{ijkl}\,\varepsilon_{kl}$$

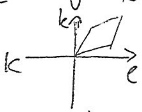
Linear Elasticity

Orthotropic symmetry, On-axis crd $\Rightarrow$

$Q_{ijkl} = Q_{klij}$

$Q_{ijkl} = Q_{jikl} = Q_{ijlk} = Q_{jilk}$

$\sigma_{ij} = \sigma_{ji}$ $\quad \varepsilon_{kl} = \varepsilon_{lk}$



$Q_{iikl} = Q_{iikl}\,\delta_{kl}$ (no sum)

$Q_{ijil} = Q_{ijil}\,\delta_{jl}$ (no sum)

9 independent components! $\Leftarrow$

Compliance $\dfrac{d\varepsilon}{d\sigma}$

$$C_{ijkl} = \frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}}$$

$$[Q]\,[C] = \mathbb{1}$$

Unit matrix

Hooke's Law $\quad \sigma = E\varepsilon \quad$ E = Young's modulus

In terms of Stiffness Components?

In terms of Compliance Components?

Spring Equation $\quad F = k\delta \quad$ k = Spring Constant

How do we get from Spring Eq. to Hooke's Law?

What is the relation between E and k?

Conductance Equation $\quad I = c\,\Delta V$

Conductivity Equation $\quad I/A_{\perp} = \gamma \dfrac{\partial V}{\partial x}$

Are these related to the above?

What is the dimension of $\begin{cases} c \\ \gamma \end{cases}$?

Resistance Equation $\quad \Delta V = RI$

Resistivity Equation $\quad \dfrac{\partial V}{\partial x} = \rho\, I/A_{\perp}$

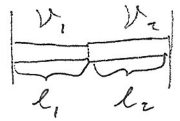
What is the dimension of $\begin{cases} c \\ \gamma \\ R \\ \rho \end{cases}$?

$$\sigma_{ij} = Q_{ijkl}\,\varepsilon_{kl}$$

How about $\begin{cases} ij \neq kl \\ i \neq j \\ \ldots \end{cases}$ in terms of conductivity?

Composite Structures

Elements in Series



$$I/A = \nu \frac{\partial V}{\partial x}$$

$$I_1 = I_2 = A_1 \nu_1 \left(\frac{\partial V}{\partial x}\right)_1 = A_2 \nu_2 \left(\frac{\partial V}{\partial x}\right)_2$$

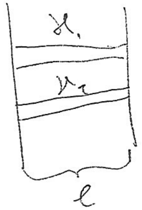
Conductivity Eq. for the Composite system:

$$I/A = \nu \frac{\Delta V}{\Delta X} = \nu \frac{\Delta V}{l_1 + l_2} = \nu \frac{\int_0^{l_1} \left(\frac{\partial V}{\partial x}\right)_1 dx + \int_0^{l_2} \left(\frac{\partial V}{\partial x}\right)_2 dx}{l_1 + l_2}$$

$$= \nu \frac{l_1 \left(\frac{\partial V}{\partial x}\right)_1 + l_2 \left(\frac{\partial V}{\partial x}\right)_2}{l_1 + l_2} = \nu \frac{l_1 \frac{I/A_1}{\nu_1} + l_2 \frac{I/A_2}{\nu_2}}{l_1 + l_2}$$

$$\Rightarrow \nu = \frac{l_1 + l_2}{\frac{\cancel{A} l_1}{\cancel{A_1} \nu_1} + \frac{A l_2}{A_2 \nu_2}}$$

Elements in Parallel



$$I/A = \nu \frac{\partial V}{\partial x}$$

$$\left(\frac{\partial V}{\partial x}\right)_1 = \left(\frac{\partial V}{\partial x}\right)_2 = \frac{\partial V}{\partial x} \Rightarrow \frac{I_1}{A_1 \nu_1} = \frac{I_2}{A_2 \nu_2}$$

$$I/A = \frac{I_1 + I_2}{A_1 + A_2} = \frac{I_1 \left(1 + \frac{A_2 \nu_2}{A_1 \nu_1}\right)}{A_1 \left(1 + \frac{A_2}{A_1}\right)} = \nu \left(\frac{\partial V}{\partial x}\right)_1 = \nu \frac{I_1}{A_1 \nu_1}$$

$$\nu = \nu_1 \frac{1 + \frac{A_2 \nu_2}{A_1 \nu_1}}{1 + \frac{A_2}{A_1}} = \frac{A_1 \nu_1 + A_2 \nu_2}{A}$$

How do we determine $\left\{\begin{array}{l}\text{stiffness}\\ \text{compliance}\end{array}\right.$ matrices experimentally?

How do we determine mechanical behavior in an arbitrary direction, once $\left\{\begin{array}{l}\text{stiffness}\\ \text{compliance}\end{array}\right.$ matrix is known in the on-axis co-ordinate system?

11.9.2007

(7)

Strain Energy Density

Differential work (scalar) $dW = F\,dS$

Differential Strain Energy Density

$$dW = \sigma_{ij}\,d\varepsilon_{ij} = \sigma_{11}\,d\varepsilon_{11} + \sigma_{12}\,d\varepsilon_{12} + \sigma_{13}\,d\varepsilon_{13} + \sigma_{21}\,d\varepsilon_{21} + \dots$$

$$\Rightarrow \quad \sigma_{ij} = \frac{dW}{d\varepsilon_{ij}}$$

$$dW = \sigma_{ij}\,d\varepsilon_{ij} = Q_{ijkl}\,\varepsilon_{kl}\,d\varepsilon_{ij} \quad \Big| \int$$

$W \approx \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$ Within $\longrightarrow$ Linear Elasticity:

$$\left(Q_{ijij} = \text{constant} \qquad \frac{d\,Q_{ijij}}{d\varepsilon_{ij}} = 0 \right)$$

12.9.2007

(8)

Strength

Strength $\equiv$ critical stress $(\sigma_{ij})_c$

Critical nominal stress ?

How about multiaxial stress states?

- We may need some multiaxial Failure Criterion.

Von Mises Stress

$$\sigma_m \equiv \frac{1}{\sqrt{2}}\sqrt{(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2)}$$

For uniaxial tension $\quad \sigma_m = \sigma_{11}$

For biaxial tension $\quad \sigma_m = \frac{1}{\sqrt{2}}\sqrt{(\sigma_{11}-\sigma_{22})^2 + \sigma_{22}^2 + \sigma_{11}^2}$

$$= \sqrt{\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{11}\sigma_{22}}$$

For Equibiaxial tension $\quad \sigma_m = \sqrt{3}\,\sigma_{ii}$

For Tension-Compression w. $\sigma_{11} = -\sigma_{22}$ $\quad \sigma_m = \sigma_{ii}$

For pure shear in one plane $\quad \sigma_m = \sqrt{3}\,\sigma_{12}$

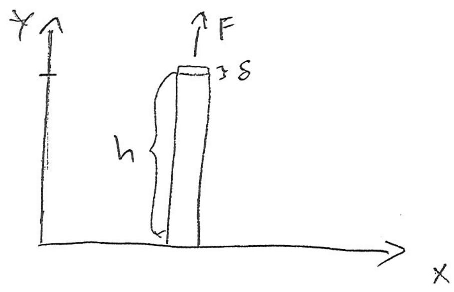
For Equitriplanar shear $\quad \sigma_m = 3\,\sigma_{ij}$

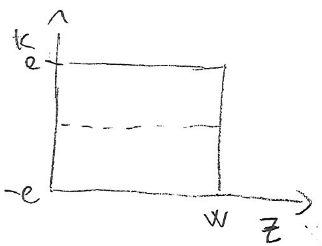
Failure Criterion $\quad \boxed{\sigma_m = \sigma_c}$ $\quad$ Right or wrong?

12.9.2007

(9)

Rigidity in Tension





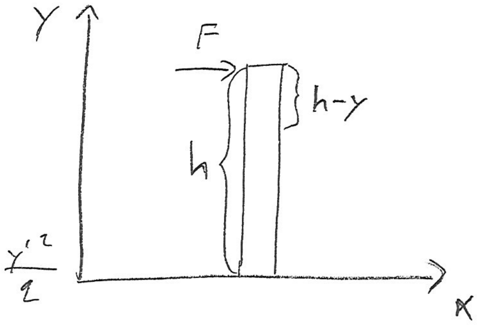
$$K \equiv \frac{F}{\delta} = \frac{\sigma A_\perp}{\varepsilon h} = \frac{E \, \sigma \int_{A_\perp} 1}{h} = \frac{E\,(2e)\,w}{h}$$

$$K' \equiv \frac{F}{\varepsilon} = E\,(2e)\,w$$

12.9.2007

(10)

Bending of a Beam



$$EI\frac{dx}{dy} = \int_0^y -M(y')\,dy'$$

$$= F\int_0^y (h-y')\,dy' = F\Big/_0^y\, hy' - \frac{y'^2}{2}$$

$$= Fy\left(h - \frac{y}{2}\right)$$

$M(y) = -F(h-y)$ for $0 \le y \le h$

$$dx = \frac{F}{EI}\left(hy - \frac{y^2}{2}\right)dy \;\Big|\; \int$$

$$\Delta x = \frac{F}{EI}\left(\frac{hy^2}{2} - \frac{y^3}{6}\right) = \frac{Fh^3}{6EI}\left(3\frac{y^2}{h^2} - \frac{y^3}{h^3}\right)$$

Curvature: $\dfrac{d^2x}{dy^2}$

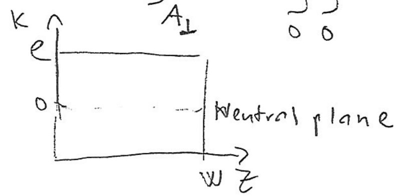
Internal External

Balance of Momenta: $EI\dfrac{d^2x}{dy^2} = -M(y)$

$$\boxed{EI\frac{d^2x}{dy^2} + M(y) = 0}$$

Second Moment of Inertia:

$$I = \int_{A_\perp} k^2 = 2\int_0^w\int_0^e k^2\,dz\,dk = 2w\frac{1}{3}e^3$$



$$\uparrow\; EI = E\frac{(2e)^3 w}{12}$$

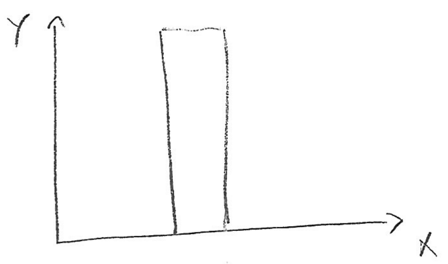
Bending Rigidity

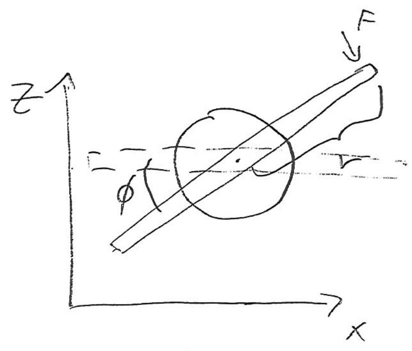
$$EI = \frac{-M(y)}{\frac{d^2x}{dy^2}} = \frac{\text{Momentum}}{\text{Curvature}}$$

2. 7. 2007

(11)

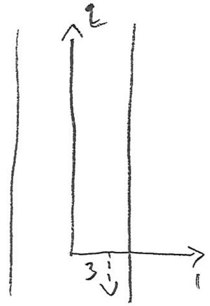
Torsional Rigidity

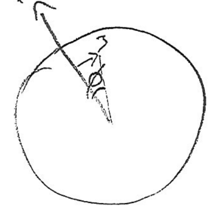


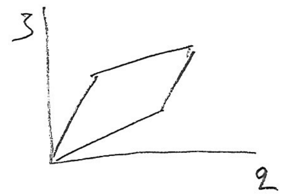


Torque $T = F \cdot r$

Torsional Rigidity $\frac{T}{\phi}$ [Nm] $=$ $\frac{\text{Torque}}{\text{Torsion}}$







$u_3 = \phi r$

$$\frac{\partial u_3}{\partial x_2} \equiv \varepsilon_{32} = \frac{\phi r}{\ell_2}$$

$$\sigma_{32} + \sigma_{23} \approx Q_{3232}\varepsilon_{32} + Q_{2323}\varepsilon_{23}$$

$$= 2Q_{3232}\varepsilon_{32} = 2Q_{3232}\frac{\phi r}{\ell_2} = \frac{dF_3}{dA_2} \qquad \frac{dF_2}{dA_3} = 0$$

$$r\frac{\partial F}{\partial A_2} = 2Q_{3232}\frac{\phi r^2}{\ell}$$

$$T = \int_{A_2} 2Q_{3232}\frac{\phi r^2}{\ell} = 2Q_{3232}\frac{\phi}{\ell}J$$

Polar Moment of Inertia

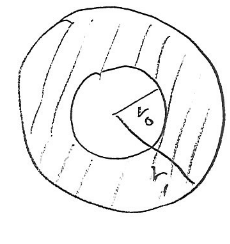
$$J = \int_A r^2$$

(12)

The Area of a Hollow Pipe

$$A = \int_0^{2\pi}\int_{r_0}^{r_1} r\,dr\,d\phi = \int_0^{2\pi} \Big/_{r_0}^{r_1} \frac{1}{2}r^2$$

$$= \pi(r_1^2 - r_0^2)$$



Polar Moment of Inertia

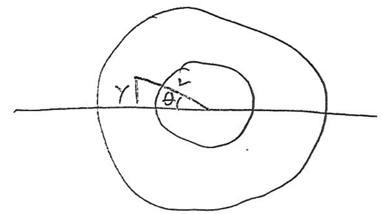
$$J = \int_0^{2\pi}\int_{r_0}^{r_1} r^3\,dr\,d\phi = \int_0^{2\pi} \Big/_{r_0}^{r_1} \frac{r^4}{4}\,d\phi = \frac{\pi}{2}(r_1^4 - r_0^4)$$

Second Moment of Inertia

$$I = \int_0^{2\pi}\int_{r_0}^{r_1} r^3 \sin^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi} \Big/_{r_0}^{r_1} \frac{r^4}{4}\sin^2\theta\,d\theta = \frac{r_1^4 - r_0^4}{4}\int_0^{2\pi} \sin^2\theta\,d\theta$$

$$= \frac{r_1^4 - r_0^4}{4} \Big/_0^{2\pi} \frac{1}{2} = \frac{\pi}{4}(r_1^4 - r_0^4)$$



$Y = r\sin\theta$

$Y^2 = r^2\sin^2\theta$

13. 9. 2007

13

Mass Density Effects

1^{st} Approximation: the amount of load-carrying material per cross-sectional area unit increases w. density

$\Rightarrow \quad Q_{ijkl} \underset{\sim}{\propto} \rho$

2^{st} Appr. for porous material $\quad P \propto \frac{1}{\rho}$

Solid Fraction $\quad S \propto \rho$

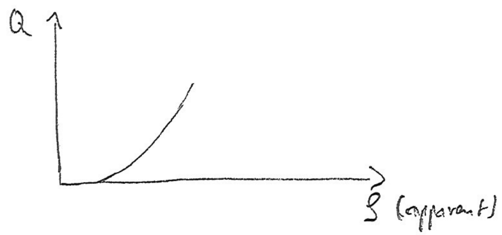
$\Rightarrow$ Connectivity of solid elements increases w. ρ

$\Rightarrow$ Specific Stiffness $\quad \frac{Q}{\rho} \propto \rho \Rightarrow Q \underset{\sim}{\propto} \rho^2$

Percolation and Connectivity

Element sparse in space do not necessarily become connected $\rightarrow$ zero stiffness at finite apparent density

Percolation: formation of a continuous network



What does the Percolation threshold depend on?

For fibers: $\frac{\text{length}}{\text{mass}} \equiv \frac{1}{\text{Coarseness}}$

13. 9. 2007

14

Trivial Scaling

Linear Dimensions scaled by q

Object scaled as $\quad q^2 \rightarrow$ Area $\qquad q^1 \rightarrow$ line

$q^3 \rightarrow$ Volume

$q^n \rightarrow$ Dimensionality n

Does the Scaling exponent have to be an Integer? Would something like $3/2$ be possible? Or $5/3$?

The object would be something between $\begin{cases} \text{line} \\ \text{Area} \end{cases}$ ... $\begin{cases} \text{Area} \\ \text{Volume} \end{cases}$ ...

$2^1 = 2$

$2^{3/2} \approx 2{,}8$ Magnification w. 2 extends "length" this much

$2^2 = 4$

$2^{5/3} \approx 5{,}7$ Magnification w. 2 extends "Area" this much...

Self-Similarity

- exact
- inexact

- w. or w.o. $\begin{cases} \text{lower} \\ \text{higher} \end{cases}$ cutoff

13. 9. 2007
15

Consequences:

Boundary Lengths, Surface Areas, and Volumes hardly exist in Nature

- only apparent values exist, and those values inherently depend on the magnification of observation

Example:

Specific Surface $\frac{A}{\rho V}$ of pulp fibers scales as (trivially)

$$\frac{A'}{\rho V'} = \frac{q^2 A}{\rho q^3 V} = q^{-1} \frac{A}{\rho V}$$

BUT The fibers do not have an area

$$q^n > q^2$$

The fibers do not have a volume

$$q^m < q^3$$

$$\frac{q^n}{q^m} = q^k > q^{-1}$$

$\boxed{\text{Is } \rho \text{ scale-invariant?}}$

17. 9. 2007
16

Size Effect on Strength

Element Failure Probability $P_1(\sigma)$

= cumulative distribution function (cdf) of strength for an Element

$\boxed{\text{pdf = probability density function}}$

Element Survival Probability

$$1 - P_1$$

Chain Survival Probability

$$1 - P_f = (1 - P_1)^N \quad \Big| \ln$$

$\boxed{\frac{d}{dp}\ln(1-p) = \frac{d(1-p)}{dp}\frac{d}{d(1-p)}\ln(1-p) = (-1)\frac{1}{1-p}}$

$$\ln(1 - P_f) = N \ln(1 - P_1)$$

$\boxed{f(x) = \sum_{n=0}^{\infty} f^{n}(0)\, x^n}$

Maclaurin Series

$$\ln(1-P_1) = \ln(1-0) + (-1)\frac{1}{1-0}P_1 + \dots$$

$$\approx -P_1$$

$$\ln(1-P_f) \approx N(-P_1)$$

$$\Rightarrow P_f \approx 1 - e^{-NP_1(\sigma)} = P_f(\sigma) \quad \text{cdf of strength for the Chain}$$

$$\left.\begin{aligned} N &\equiv \frac{V}{V_c} \\ \frac{1}{V_c}P_1(\sigma) &\equiv c(\sigma) \end{aligned}\right\} \Rightarrow \boxed{P_f(\sigma, V) = 1 - e^{-c(\sigma)V}}$$

Weibull 1939: $c(\sigma) \cong \frac{1}{V_0}\left\langle\frac{\sigma - \sigma_1}{\sigma_0}\right\rangle^m$

$\langle \text{abs}\, x \rangle = x$

$\langle -\text{abs}\, x \rangle = 0$

$$\Rightarrow \boxed{P_f(\sigma, V) = 1 - e^{-\frac{V}{V_0}\left\langle\frac{\sigma-\sigma_1}{\sigma_0}\right\rangle^m}} \rightarrow 1 - e^{-\frac{V}{V_0}\left\langle\frac{\sigma}{\sigma_0}\right\rangle^m} \text{ for } \sigma_1 = 0$$

17.7.2007

(17)

What is the pdf of strength?

$$\frac{d}{d\sigma} P_f = \frac{d\frac{\sigma-\sigma_1}{\sigma_0}}{d\sigma} \cdot \frac{d\left(\frac{\sigma-\sigma_1}{\sigma_0}\right)^m}{d\frac{\sigma-\sigma_1}{\sigma_0}} \frac{d}{d\left(\frac{\sigma-\sigma_1}{\sigma_0}\right)^m} P_f$$

$$= \frac{1}{\sigma_0} m \left(\frac{\sigma-\sigma_1}{\sigma_0}\right)^{m-1} \frac{V}{V_0} e^{-\frac{V}{V_0}\left(\frac{\sigma-\sigma_1}{\sigma_0}\right)^m} = p_f$$

What is the mean value of strength?

$$\bar{\sigma} = \int_{\sigma_1}^{\infty} \sigma\, p_f\, d\sigma \quad \rightarrow \text{somewhat complicated to integrate}$$

$$\boxed{\frac{dP_f}{d\sigma} = p_f \implies p_f\, d\sigma = dP_f}$$

$$= \int_0^1 \sigma\, dP_f$$

still difficult...

What is the median value of strength?

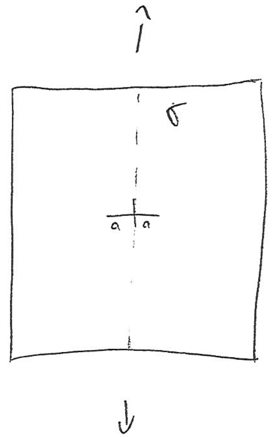
$$P_f(\sigma, V) = 0{,}5 \implies e^{-\frac{V}{V_0}\left(\frac{\sigma-\sigma_1}{\sigma_0}\right)^m} = 0{,}5$$

$$\implies \frac{V}{V_0}\left(\frac{\sigma-\sigma_1}{\sigma_0}\right)^m = \ln 2 \implies \sigma_{0,5} = \sigma_0 \left(\frac{V_0}{V}\ln 2\right)^{\frac{1}{m}} + \sigma_1$$

18.7.2007

(18)

Fracture Mechanics Size Effect



Failure criterion

$$\underset{\leq 0}{\frac{d\Pi}{da}} + \underset{\geq 0}{\frac{dW_s}{da}} \leq 0$$

$$\frac{dW_s}{da} = \frac{d\, 4at\mathcal{G}}{da} = 4t\mathcal{G}$$

$$A \equiv 4ta$$

$$\frac{dW_s}{dA} = \mathcal{G}$$

Inglis: $\Pi = \Pi_0 - \frac{\pi\sigma^2 a^2 t}{Q}$

$$\frac{d\Pi}{da} = -\frac{2\pi\sigma^2 a t}{Q}$$

Criticality in LEFM

$$\frac{d\Pi}{da} + \frac{dW_s}{da} = 0$$

$$\frac{dW_s}{da} = -\frac{d\Pi}{da}$$

$$4t\mathcal{G} = \frac{2\pi\sigma^2 a t}{Q}$$

$$\mathcal{G} = \frac{\pi\sigma^2 a}{2Q} \implies \sigma_c = \sqrt{\frac{2Q\mathcal{G}}{\pi a}}$$

LEFM Size Effect

$$\boxed{\sigma_c \propto D^{-\frac{1}{2}}}$$

How about material w. plastic yielding at σ_{pe}?

Let us try a scaling parameter $\beta = \frac{\sigma_{pe}^2}{\sigma_c^2} = \frac{\pi a \sigma_{pe}^2}{2Q\mathcal{G}}$

w. strength scaling $\sigma_c = \frac{\sigma_{pe}}{\sqrt{1 + \frac{\pi a \sigma_{pe}^2}{2Q\mathcal{G}}}} = \frac{\sigma_{pe}}{\sqrt{1+\beta}} = \frac{1}{\sqrt{\frac{1}{\sigma_{pe}^2} + \frac{1}{\sigma_c^2}}}$

$$\boxed{\sigma_c \propto (1+\beta)^{-\frac{1}{2}} = \left(1 + \frac{D}{\ell_{ch}}\right)^{-\frac{1}{2}}}$$

Size-Effect Scaling

$$D \equiv \frac{\pi a}{2} \qquad \ell_{ch} \equiv \frac{Q\mathcal{G}}{\sigma_{pe}^2}$$

25.8.2007 (19)

Surface Energy $\quad \gamma A$

$$F = \frac{dW}{d\delta} = \frac{d\,\gamma A}{dr} = \frac{d\,\gamma\, 4\pi r^2}{dr} = 8\pi r \gamma$$

Stress due to surface tension $\quad \frac{F}{A} = \frac{8\pi r\gamma}{4\pi r^2} = \frac{2\gamma}{r}$

Balance of Forces

$$p_{in}\, 4\pi r^2 = p_{out}\, 4\pi r^2 + 8\pi r\gamma$$

$$\Delta p = \frac{2\gamma}{r} \qquad \text{Laplace Eq.}$$

Internal Energy

$U = TS - pV + \mu N \qquad dU = T\,dS - p\,dV + \mu\,dN$

Gibbs Function

$$G = U - TS + pV \qquad dG = -S\,dT + V\,dp + \mu\,dN$$

$$= \mu N \qquad = \frac{\partial G}{\partial T}dT + \frac{\partial G}{\partial p}dp + \frac{\partial G}{\partial N}dN$$

$$\Rightarrow \mu = \frac{G}{N}$$

Molar Gibbs Function

$$G_m = \frac{G}{n} = \mu N_A \qquad dG_m = -\frac{S}{n}dT + \frac{V}{n}dp + \frac{\mu}{n}dN$$

$$= -S_m\,dT + V_m\,dp + \mu_m\,dN$$

Equil.

at dT

Gas

J Eq.

25.8.2007 (20)

Equilibrium

$$\mu_g = \mu_e \Rightarrow G_{mg} = G_{me}$$

at $dT = 0$ $\qquad V_{mg}\,dp_g = V_{me}\,dp_e = V_{me}\,(dp(r=\infty) + d\Delta p)$

Ideal Gas $\quad pV_m = RT$

$$\frac{RT}{p_g}dp_g = V_{me}\,dp_\infty + V_{me}\,d\Delta p$$

$$RT \ln p_g = V_{me}\,p_\infty + V_{me}\,\Delta p + C$$

$$p_g = e^{\frac{V_{me}p_\infty}{RT}}\, e^{\frac{V_{me}2\gamma}{rRT}}\, C_3 \qquad r \to \infty \Rightarrow p_g \to p_\infty$$

$$= C_4\, e^{\frac{V_{me}2\gamma}{rRT}} = p_\infty\, e^{\frac{2\gamma M}{rRT\rho}} \qquad \text{Kelvin Eq.}$$

Set $p_g(r = r_s) = p_s$

$$p_s = p\, e^{\frac{2\gamma M}{r_s RT\rho}} \Rightarrow \frac{p}{p_s} = e^{-\frac{2\gamma M}{r_s RT\rho}}$$

Expansion due to Isotropic pressure

$$\Delta p \rightarrow \quad \sigma'_{11} = \sigma'_{22} = \sigma'_{33} = p$$

$$\varepsilon'_{ij} = C_{ijkl}\,\sigma'_{kl}$$

19.9.2007 (21)

How do we invert a matrix?
by Gaussian Elimination

Linear transformation $A\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x'\\y'\end{pmatrix}$ (1)

$$A^{-1}A\begin{pmatrix}x\\y\end{pmatrix} = A^{-1}\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} \quad (3)$$

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$$

Original transformation

$$\begin{matrix}1a)\\b)\end{matrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x'\\y'\end{pmatrix} \Leftrightarrow$$

1a) $ax + by = x'$

1b) $cx + by = y' \quad |\cdot -\frac{a}{c}$

2b) $\left(b - \frac{ab}{c}\right)y = x' - \frac{a}{c}y'$

$$\begin{matrix}1)\\b)\end{matrix}\begin{pmatrix}a&b\\0&b-\frac{ab}{c}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1&0\\1&-\frac{a}{c}\end{pmatrix}\begin{pmatrix}x'\\y'\end{pmatrix}$$

1a) $ax + by = x' \quad \left|\cdot\left(-\frac{b-\frac{ab}{c}}{b}\right)\right.$

2b) $\left(b-\frac{ab}{c}\right)y = x' - \frac{a}{c}y'$

2a) $-\frac{a}{b}\left[b-\frac{ab}{c}\right]x = \frac{a}{c}x' - \frac{a}{c}y'$

$$\begin{matrix}2)\\2)\end{matrix}\begin{pmatrix}-\frac{a}{b}\left[b-\frac{ab}{c}\right]&0\\0&b-\frac{ab}{c}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{a}{c}&-\frac{a}{c}\\1&-\frac{a}{c}\end{pmatrix}\begin{pmatrix}x'\\y'\end{pmatrix}$$

Inverted Transformation

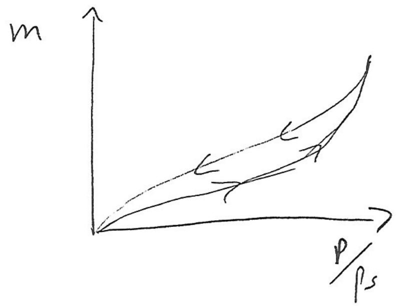
$$\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-\frac{b}{c}\frac{1}{b-\frac{ab}{c}} & \frac{b}{c}\frac{1}{b-\frac{ab}{c}}\\ \frac{1}{b-\frac{ab}{c}} & -\frac{a}{c}\frac{1}{b-\frac{ab}{c}}\end{pmatrix}\begin{pmatrix}x'\\y'\end{pmatrix}$$

$$= A^{-1}\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} \quad (3)$$

How can we check the Inversion is correct?

24.9.2007 (22)

ADSORPTION HYSTERESIS

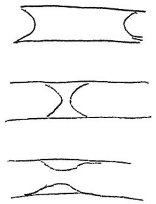


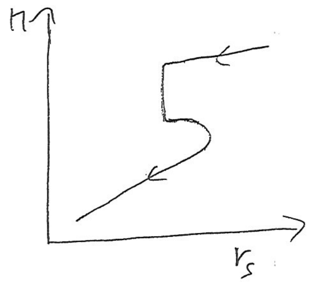
Why does Equilibrium moisture content depend on history?

Kelvin Eq. $\frac{p}{p_s} = e^{-\frac{\gamma 2M}{r_s \rho R T}} \Rightarrow -\ln\frac{p}{p_s} \propto \frac{1}{r_s}$

$r_s \propto \frac{1}{-\ln p/p_s}$

Desorption of water from a capillary:





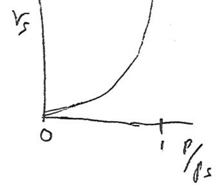
p: $0 \to p_s$

$\frac{p}{p_s}$: $0 \to 1$

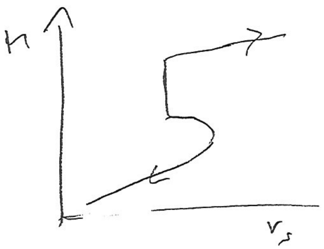
$\ln\frac{p}{p_s}$: $-\infty \to 0$

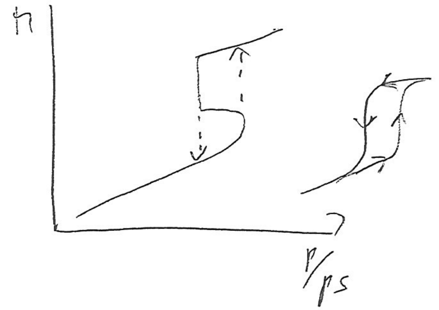
$-\ln p/p_s$: $\infty \to 0$

$\frac{1}{-\ln p/p_s}$: $0 \to \infty$



Adsorption

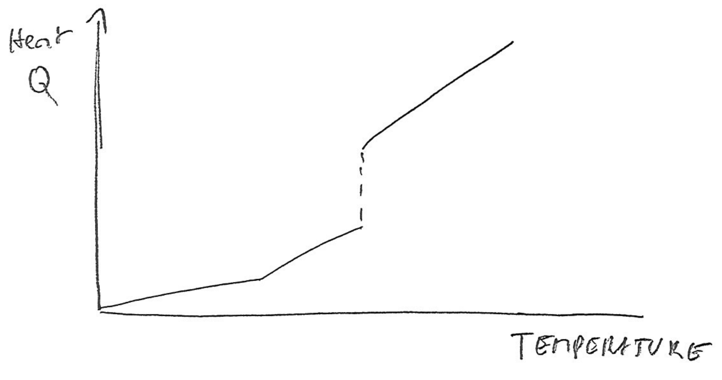




24.7.07

(23)

THERMAL TRANSITIONS



Heat Capacity $\frac{dQ}{dT}$

First-Order transition

Second-Order transition

Melting Temperature Spectrum

Clausius-Clapeyron $\frac{dp}{dT} = \frac{\Delta H}{T \Delta V} < 0$

$dp = \frac{\Delta H}{T \Delta V} dT$

$p = \frac{\Delta H}{\Delta V} \ln T + C$

$T = C_2 e^{p\frac{\Delta V}{\Delta H}} = C_2 e^{-p\frac{\Delta V}{\Delta H}}$

Kelvin Eq. II

$p = p_\infty e^{\frac{\gamma 2M}{r R \delta T}}$

$T_m = C_2 e^{-p_\infty e^{\frac{\gamma 2M}{r R \delta T}} \left|\frac{\Delta V}{\Delta H}\right|}$

$T_0 \equiv T(r = \infty) \equiv C_2 e^{-p_\infty \left|\frac{\Delta V}{\Delta H}\right|}$

$\frac{T_m}{T_0} = e^{p_\infty \left|\frac{\Delta V}{\Delta H}\right| \left(1 - e^{\frac{\gamma 2M}{r R \delta T}}\right)}$

$\ln \frac{T_m}{T_0} = p_\infty \left|\frac{\Delta V}{\Delta H}\right| \left(1 - e^{\frac{\gamma 2M}{r R \delta T}}\right)$

$\frac{\gamma 2M}{r R \delta T} = \ln\left(1 - \frac{\ln \frac{T_m}{T_0}}{p_\infty \left|\frac{\Delta V}{\Delta H}\right|}\right)$

$r = \frac{\gamma 2M}{\dots} \left(1 - \ln \frac{T_m}{T_0} \dots\right)$

w. some simplifying assumptions

$r = -\frac{2\gamma M}{\Delta H_m \delta \ln\left(\frac{T_m}{T_0}\right)}$ Here $\gamma \approx 12 \cdot 10^{-3} \frac{N}{m}$

1.10.2007

(24)

Divergence Theorem (Gauss)

$$\int_V \nabla \cdot \bar{a} \, dV = \oint_S \bar{a} \cdot d\bar{S} = \int_S \bar{a} \cdot \hat{n} \, dS$$

$\nabla = \hat{e}_i \frac{\partial}{\partial x_i}$ $\nabla \cdot \bar{a} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}$

$\bar{a} = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3$

Fick's First Law

Flux $\bar{J} = -[D] \nabla \cdot \phi$ $\phi = \frac{\partial Q}{\partial V}$

$J_i = \frac{\partial^2 Q}{\partial t \, \partial A_{\perp i}}$ $[D]$ = Diffusivity matrix

$$\begin{pmatrix} J_1 \hat{e}_1 \\ J_2 \hat{e}_2 \\ J_3 \hat{e}_3 \end{pmatrix} = \begin{pmatrix} \bar{J}_1 \\ \bar{J}_2 \\ \bar{J}_3 \end{pmatrix} = -\begin{pmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{pmatrix} \begin{pmatrix} \hat{e}_1 \frac{\partial \phi}{\partial x_1} \\ \hat{e}_2 \frac{\partial \phi}{\partial x_2} \\ \hat{e}_3 \frac{\partial \phi}{\partial x_3} \end{pmatrix}$$

Total Flow ~~into~~ out of Volume V

$-\frac{dQ}{dt} = \oint_S \bar{J} \cdot d\bar{S} = \oint_S -[D] \nabla \cdot \phi \cdot d\bar{S} = \int_V \nabla \cdot \bar{J} \, dV$

$\rightarrow \int_V -[D] \nabla^2 \phi \, dV$

$Q = \int_V \phi \, dV$

$-\frac{dQ}{dt} = \int_V -\frac{d\phi}{dt}$

$\boxed{\frac{d\phi}{dt} = [D] \nabla^2 \phi}$ Diffusion Eq.

$\boxed{\frac{d\phi}{dt} + \nabla \cdot \bar{J} = 0}$ Continuity Eq.

1.10.2007

(25)

Thermal Flux Eq.

$$J_i = -D_i \frac{\partial}{\partial x_i} \phi$$

$$\frac{\partial^2 Q}{\partial t\, \partial A_{\perp i}} = -D_i \frac{\partial^2 Q}{\partial x_i\, \partial V}$$

$$\left[\frac{J}{s\,m^2}\right] \quad \left[\frac{m^2}{s}\right] \left[\frac{J}{m^4}\right]$$

Thermal Diffusivity

$$D_i = -\frac{\partial x_i\, \partial V}{\partial t\, \partial A_{\perp i}} = -\frac{x_i\, \partial x_i}{\partial t}$$

How about Temperature Gradient as Flux Driving Factor?

Thermal Conductivity Eq.

$$J_i = -k_i \frac{\partial}{\partial x_i} T$$

$$\frac{\partial^2 Q}{\partial t\, \partial A_{\perp i}} = -k_i \frac{\partial}{\partial x_i} T$$

$$\left[\frac{J}{s\,m^2}\right] \quad \left[\frac{J}{m\,s\,K}\right] \left[\frac{K}{m}\right]$$

Thermal Conductivity

$$k_i = -\frac{\partial^2 Q\, \partial x_i}{\partial t\, \partial A_{\perp i}\, \partial T}$$

$$\frac{k_i}{D_i} = \frac{\partial^2 Q\, \partial x_i}{\partial t\, \partial A_{\perp i}\, \partial T} \frac{\partial t\, \partial A_{\perp i}}{\partial x_i\, \partial V} = \frac{\partial^2 Q}{\partial T\, \partial V}$$

= Volumetric Heat Capacity

$= C_v$

$$k_i = C_v D_i$$

1.10.2007

(26)

How do we use the Diffusion Eq.?

Steady-state problems $\rightarrow \frac{d\phi}{dt} = 0 \Rightarrow$ Laplace Eq.

Transient problems: Fourier series solution

Example: $\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$

Heater, $\frac{\text{power}}{\text{Area}} = H$ $\boxed{J(L,t) = -H}$

Wall of thickness $= L$

$$\left.\begin{array}{ll} \text{Outside temperature} & u(0,t) = 0 \\ \text{Initial temperature} & u(x,0) = 0 \\ \text{Heat Flux at source} & -k \frac{\partial u}{\partial x} = H \end{array}\right\} \begin{array}{l}\text{Boundary Conditions} \\ \text{Inhomog.}\end{array}$$

Potential function transformation: $u(x,t) = v(x,t) + w(x)$

$$D\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right) = \frac{\partial v}{\partial t}$$

$$v(x,0) + w(x) = 0$$

$$v(0,t) + w(0) = 0$$

$$\frac{\partial v(L,t)}{\partial x} + \frac{\partial w(L)}{\partial x} = \frac{H}{k} \qquad \text{set } w(x) \equiv \frac{Hx}{k}$$

$$\left.\begin{array}{l} \Rightarrow \frac{\partial v(L,t)}{\partial x} = 0 \\ \frac{\partial v}{\partial x} = -\frac{H}{k} \Rightarrow v(x,0) = -\frac{Hx}{k} \\ v(0,t) = 0 \end{array}\right\} \begin{array}{l}\text{Homogeneous} \\ \text{Boundary conditions}\end{array}$$

Now separate $v(x,t) = X(x)\,T(t)$

$$D \frac{\partial^2 (X(x)\,T(t))}{\partial x^2} = \frac{\partial X(x)\,T(t)}{\partial t}$$

$$D X''T = XT' \Rightarrow \frac{X''}{X} = \frac{T'}{T}\frac{1}{D} \equiv -\lambda^2$$

2.10.2007

27

$X'' = -\lambda^2 X \quad \Rightarrow X = a e^{i\lambda x} + b e^{-i\lambda x}$

$= A\cos\lambda x + B\sin\lambda x$

$T' = -\lambda^2 D T$

$\Rightarrow T = c \cdot e^{-\lambda^2 D t}$

$v(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\lambda^2 D t}$

$v(0,t) = 0 \Rightarrow A = 0$

$v(x,t) = B\sin(\lambda x)\, e^{-\lambda^2 D t}$

$\frac{\partial v(L,t)}{\partial x} = 0 \quad \Rightarrow \quad B\lambda\cos(\lambda L) = 0$

$\Rightarrow \lambda = \frac{n\pi}{2L}, \quad n = 1,3,5,\ldots$

One Solution:

$v(x,t) = B\sin\left(\frac{n\pi}{2L}x\right) e^{-\frac{n^2\pi^2}{4L^2}Dt}$

Superposition of Solutions, Last Boundary Condition:
(For $t=\infty$)

$\sum_{n_{odd}} B_n \sin\left(\frac{n\pi}{2L}x\right) = -\frac{Hx}{K}$

Fourier Sine Series $L' = 4L$

$f(x) = B_n \sum_{L'} \sin\left(\frac{n\pi}{2L}x\right) = B_n \sum \sin\left(\frac{2\pi}{L'}\right)$

$B_n = \frac{2}{L'}\int f(x)\sin\left(\frac{2\pi n}{L'}x\right)dx$

$\Rightarrow$ Fourier Series Coefficients $\quad B_n = -\frac{8HL}{K\pi^2}\frac{(-1)^{(n-1)/2}}{n^2}$

$u(x,t) = v(x,t) + w(x)$

$= \frac{Hx}{K} - \frac{8H}{K\pi^2}\sum_{n_{odd}}\frac{(-1)^{(n-1)/2}}{n^2}\sin\left(\frac{n\pi x}{2L}\right)\exp\left(-\frac{Dn^2\pi^2}{4L^2}t\right)$

8.10.2007

28

Fick's Law

$\frac{\partial^2 Q}{\partial t\,\partial A_\perp} = -D\frac{\partial^2 Q}{\partial x\,\partial V}$

$= -D\frac{\partial \phi}{\partial x}$

$\sigma_{12} = \mu\dot{\varepsilon}_{12}$

$\left[\frac{N}{m^2}\right] \quad \left[\frac{1}{s}\right]$

Darcy's Law

$\frac{\partial^2 V}{\partial t\,\partial A_\perp} = -\frac{K}{\mu}\frac{\partial P}{\partial x}$

μ = viscosity $\left[\frac{Ns}{m^2}\right]$

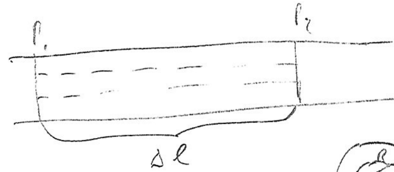
$\frac{m}{s} \qquad \frac{\quad}{\left[\frac{Ns}{m^2}\right]} \left[\frac{N}{m^3}\right]$

K = permeability $\left[m^2\right]$

In 3-d form

$\bar{J} = -\frac{[K]}{\mu}\nabla P$

Hagen - Poisenlle Law



$-(P_2-P_1)\pi r^2 + \sigma_{er}\, 2\pi r\,\Delta l = 0$

$-(P_2-P_1)\pi r^2 + \mu\dot{\varepsilon}_{er}\, 2\pi r\,\Delta l = 0$

$\dot{\varepsilon}_{er} = \frac{d}{dt}\frac{du}{dr} = \frac{d}{dr}\frac{du}{dt} = \frac{dv}{dr}$

$\dot{\varepsilon}_{er} = \frac{(P_2-P_1)r}{\mu\, 2\,\Delta l} = \frac{dp}{dl}\frac{r}{2\mu}$

$\frac{dv}{dr} = \frac{dp}{dl}\frac{r}{2\mu}$

$dv = \frac{dp}{dl}\frac{r}{2\mu}dr$

$v = \frac{dp}{dl}\frac{r^2}{4\mu} + C$

$v(r=R) = 0 \Rightarrow C = -\frac{dp}{dl}\frac{R^2}{4\mu}$

$v(r) = \frac{dp}{dl}\frac{r^2 - R^2}{4\mu}$

$\frac{dV}{dt} = \int_0^R 2\pi r\, v(r)\, dr =$

$-\frac{dp}{dl}\frac{2\pi}{4\mu}\int_0^R R^2 r - r^3\, dr = -\frac{\pi}{2\mu}\frac{dp}{dl}\left|\frac{R^2 r^2}{2} - \frac{r^4}{4}\right|_0^R$

$= -\frac{\pi}{8\mu}\frac{dp}{dl}R^4$

$\frac{\partial^2 V}{\partial t\,\partial A_\perp} = \frac{\partial}{\partial \pi R^2}\frac{\partial V}{\partial t} = \frac{\partial}{\partial R^2} - \frac{dp}{dl}\frac{(R^2)^2}{8\mu} = -\frac{R^2}{4\mu}$

$\Rightarrow K = \frac{R^2}{4}$ (?)

8.10.2007

29

Porosity Effect

$$\frac{\partial}{\partial A_\perp}\frac{\partial V}{\partial t} = \frac{\partial N}{\partial A}\frac{\partial}{\partial N}\frac{\partial V}{\partial t} = \frac{\mathbb{P}}{\pi R^2}\left[-\frac{\pi}{8\mu}\frac{dp}{d\ell}R^4\right] = -\mathbb{P}\,\frac{R^2}{8\mu}\frac{dp}{d\ell}$$

$$\mathbb{P}A = N\pi R^2 \Rightarrow N = \frac{\mathbb{P}A}{\pi R^2} \Rightarrow \frac{\partial N}{\partial A} = \frac{\mathbb{P}}{\pi R^2} \qquad \Rightarrow K = \mathbb{P}\frac{R^2}{8}$$

Variable Size of pores

$$K = \int \mathbb{P}\,\frac{R^2}{8}\,p(R)\,dR$$

$$\int p(R)\,dR = 1$$

Variable Pore Geometry, Alignment

$$K \propto \mathbb{P}\,R_{ch}^2$$

10.10.2007

30

Time-Dependent Mechanical Behavior
- Linear Viscoelasticity

$$\varepsilon_{ij} = C_{ijkl}\,\sigma_{kl} \rightarrow d\varepsilon_{ij}(t) = C_{ijkl}(t-\tau)\,d\sigma_{kl}(\tau)$$

$$\varepsilon_{ij}(t) = \int_{-\infty}^{t} C_{ijkl}(t-\tau)\,\frac{d\sigma_{kl}(\tau)}{d\tau}\,d\tau$$

$C_{ijkl}(t) \equiv$ Creep Compliance

$$\sigma_{ij} = Q_{ijkl}\,\varepsilon_{kl} \rightarrow d\sigma_{ij}(t) = Q_{ijkl}(t-\tau)\,d\varepsilon_{kl}(\tau)$$

$$\sigma_{ij} = \int_{-\infty}^{t} Q_{ijkl}(t-\tau)\,\frac{d\varepsilon_{kl}(\tau)}{d\tau}\,d\tau$$

$Q_{ijkl}(t) \equiv$ Relaxation Modulus

Thermorheologically Simple Materials

$$C(T,t) = C\left(T_0, t/a(T)\right) \qquad Q(T,t) = Q\left(T_0, t/b(T)\right)$$

$a \approx b$ (?)

WLF

$$\log a = -\frac{c_1(T-T_0)}{c_2 + T - T_0} \qquad \log a = \frac{\ln a}{\ln 10}$$

$$a = 10^{-\frac{c_1(T-T_0)}{c_2+T-T_0}}$$

Reduced time $\equiv t/a(T)$

$T > T_0 \Rightarrow a < 1$

$T < T_0 \Rightarrow a > 1$

10.10.2007

(31)

What kind of a Function is the Relaxation Modulus?

Maxwell Element

$$\sigma = Q\varepsilon_1 = \eta\dot{\varepsilon}_2$$

$$\varepsilon = \varepsilon_1 + \varepsilon_2$$

$$\frac{d\varepsilon_1}{dt} + \frac{d\varepsilon_2}{dt} = 0$$

$$\frac{1}{Q}\frac{d\sigma}{dt} + \frac{\sigma}{\eta} = 0$$

$$\frac{1}{Q}\frac{d\sigma}{dt} = -\frac{\sigma}{\eta}$$

$$\frac{d\sigma}{\sigma} = -\frac{Q}{\eta}dt$$

$$\ln\sigma = -\frac{Q}{\eta}t + C$$

$$\sigma = C_1 e^{-t/\tau} = Q_0\varepsilon e^{-t/\tau}$$

$$Q(t) = Q_0 e^{-t/\tau}$$

Maxwell Elements in Parallel

$$\sigma = \sum_i \sigma_i = \sum_i Q_i(t)\,\varepsilon$$

$$= \varepsilon\sum_i (Q_0)_i\, e^{-t/\tau_i}$$

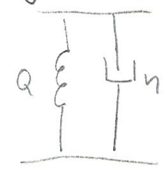
$$Q(t) = \int_i (Q_0)_i\, e^{-t/\tau_i}\, di = \int_0^\infty Q_0(\tau)\, e^{-t/\tau}\frac{di}{d\tau}\, d\tau$$

10.10.2007

(32)

What kind of a function is the Creep Compliance?

Voigt Element



$$\sigma = \varepsilon Q + \dot{\varepsilon}\eta$$

$$\frac{d\sigma}{dt} = 0 \Rightarrow \dot{\varepsilon}Q + \ddot{\varepsilon}\eta = 0$$

merk. $\dot{\varepsilon} = a$

$$aQ + \dot{a}\eta = 0$$

$$aQ = -\frac{da}{dt}\eta$$

$$-\frac{Q}{\eta}dt = da$$

$$a = Ce^{-\frac{Q}{\eta}t} \simeq \frac{\sigma}{\eta}e^{-\frac{Q}{\eta}t}$$

$$\frac{d\varepsilon}{dt} = \frac{\sigma}{\eta}e^{-\frac{Q}{\eta}t}$$

$$\varepsilon = \frac{\sigma}{\eta}\left(-\frac{\eta}{Q}\right)e^{-\frac{Q}{\eta}t} + C$$

$$= -\frac{\sigma}{Q}e^{-\frac{Q}{\eta}t} + \frac{\sigma}{Q} = \frac{1}{Q}\left(1 - e^{-\frac{Q}{\eta}t}\right)\sigma$$

$$C = C_\infty\left(1 - e^{-\frac{Q}{\eta}t}\right) = C_\infty\left(1 - e^{-\frac{t}{\tau}}\right)$$

Voigt Elements in Series

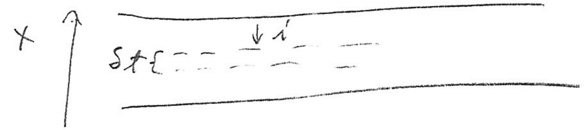
$$\varepsilon = \sum_i (C_\infty)_i\left(1 - e^{-t/\tau_i}\right)\sigma$$

$$C(t) = \int_0^\infty (C_\infty)_i\left(1 - e^{-t/\tau_i}\right) di = \int_0^\infty C_\infty(\tau)\left(1 - e^{-t/\tau}\right)\frac{di}{d\tau}\, d\tau$$

15.10.2007

(33)

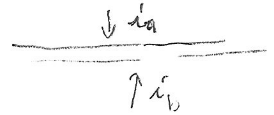
Kubelka-Munk Optics



$$\frac{di_R/i}{dx} \equiv S \qquad \frac{di_A/i}{dx} \equiv K$$

$$i = i_R + i_T + i_A$$

$$1 = \frac{i_R}{i} + \frac{i_T}{i} + \frac{i_A}{i}$$



$$-di_a = -(S+K)\,i_a\,dx + S\,i_b\,dx$$

$$di_b = -(S+K)\,i_b\,dx + S\,i_a\,dx$$

$$\frac{di_b}{i_b} - \frac{di_a}{i_a} = -2(S+K)\,dx + S\left(\frac{i_b}{i_a} + \frac{i_a}{i_b}\right) dx \qquad \frac{dx}{x} = d\ln x$$

$$-d\ln r = -2(S+K)\,dx + S\left(r + \frac{1}{r}\right) dx \qquad \frac{i_a}{i_b} \equiv r \qquad \ln a - \ln b = \ln\frac{a}{b}$$

$$= -\frac{dr}{r}$$

$$-dr = -2(S+K)\,r\,dx + S r^2 dx + S dx$$

$$-S\,dx = \frac{dr}{r^2 - 2r\frac{S+K}{S} + 1} \equiv \frac{dr}{r^2 - 2ar + 1} = \frac{dr}{\left(r-a+\sqrt{a^2-1}\right)\left(r-a-\sqrt{a^2-1}\right)}$$

$$= \frac{dr}{2\sqrt{a^2-1}}\left[\frac{-1}{r-a+\sqrt{a^2-1}} + \frac{1}{r-a-\sqrt{a^2-1}}\right]$$

Partial Fractions Decomposition

$$\frac{1}{\left(r-a+\sqrt{a^2-1}\right)\left(r-a-\sqrt{a^2-1}\right)} = \frac{A}{r-a+\sqrt{a^2-1}} + \frac{B}{r-a-\sqrt{a^2-1}}$$

$$1 = A\left[r-a-\sqrt{a^2-1}\right] + B\left[r-a+\sqrt{a^2-1}\right]$$

$$= (A+B)(r-a) + (B-A)\sqrt{a^2-1}$$

$1 =$ constant for any $r \Rightarrow A+B=0$

$$\Rightarrow A = -B \Rightarrow 1 = 2B\sqrt{a^2-1}$$

$$\Rightarrow B = \frac{1}{2\sqrt{a^2-1}}$$

15.10.2007

(34)

$$2S\sqrt{a^2-1}\,dx = \left[\frac{1}{r-a+\sqrt{a^2-1}} - \frac{1}{r-a-\sqrt{a^2-1}}\right] dr$$

$$\int_0^X 2S\sqrt{a^2-1}\,dx = \int_{R_0}^{R_X} [\;]\,dr$$

$$SX\sqrt{a^2-1} = \Big/_{R_0}^{R_X} \ln\frac{r-a+\sqrt{a^2-1}}{r-a-\sqrt{a^2-1}}$$

$$2SX\sqrt{a^2-1} = \ln\frac{R_0-a-\sqrt{a^2-1}}{R_0-a+\sqrt{a^2-1}}\,\frac{R_X-a+\sqrt{a^2-1}}{R_X-a-\sqrt{a^2+1}} \qquad r \equiv \frac{i_a}{i_b}$$

$$\boxed{X \to \infty \Rightarrow R_0 \to R_\infty,\quad R_\infty - \left(a-\sqrt{a^2-1}\right) \to 0}$$

$$\Rightarrow R_\infty = a - \sqrt{a^2+1} = 1 + \frac{K}{S} - \sqrt{\left(1+\frac{K}{S}\right)^2 - 1} = 1 + \frac{K}{S} - \sqrt{\left(\frac{K}{S}\right)^2 + 2\frac{K}{S}}\ \%$$

Limit Reflectivity

$$2SX\sqrt{a^2-1} = \ln\left(\frac{R_0+R_\infty-2a}{R_0-R_\infty}\,\frac{R_X-R_\infty}{R_X+R_\infty-2a}\right)$$

Finite thickness of sheet w. non-reflecting background $\Rightarrow R(X)=0$

$$2SX\sqrt{a^2-1} = \ln\left(\frac{R_0+R_\infty-2a}{R_0-R_\infty}\,\frac{-R_\infty}{R_\infty-2a}\right)$$

$$R_\infty = a - \sqrt{a^2-1}$$

$$\frac{1}{R_\infty} = \frac{1}{a-\sqrt{a^2-1}} = \frac{a+\sqrt{a^2-1}}{\cancel{a^2}-\cancel{a^2}+1}$$

$$\frac{1}{R_\infty} + R_\infty = 2a$$

$$\frac{1}{R_\infty} - R_\infty = 2\sqrt{a^2-1}$$

$$R_\infty^2\,\frac{R_0 - \frac{1}{R_\infty}}{R_0 - R_\infty} = \exp\left[SX\left(\frac{1}{R_\infty} - R_\infty\right)\right]$$

$$R_0 - \frac{1}{R_\infty} = \frac{R_0 - R_\infty}{R_\infty^2}\exp[\;]$$

$$R_0\left(1 - \frac{\exp[\;]}{R_\infty^2}\right) = -\frac{\exp[\;]}{R_\infty} + \frac{1}{R_\infty}$$

$$R_0 = \frac{1}{R_\infty}\,\frac{1-\exp\left[SX\left(\frac{1}{R_\infty}-R_\infty\right)\right]}{1-\frac{\exp[\;]}{R_\infty^2}} = R_\infty\,\frac{\exp\left[SX\left(\frac{1}{R_\infty}-R_\infty\right)\right]-1}{\exp\left[SX\left(\frac{1}{R_\infty}-R_\infty\right)\right]-R_\infty^2}$$

Reflectivity

$$\frac{R_0}{R_\infty} \equiv \text{Opacity}$$

Partial Fractions Decomposition

$$f(x) = \frac{g(x)}{h(x)}$$

The ratio of polynomials changes its value most rapidly in the vicinity of argument values $x = \alpha$, where $h(\alpha) = 0$.

$\Rightarrow$

$$f(x) = \frac{A_1}{(x-\alpha_1)^{n_1}} + \frac{A_2}{(x-\alpha_2)^{n_2}} + \dots$$

1° How many terms should this decomposition have?
— As many as there are distinct roots for $h(\alpha) = 0$

2° What should be the exponents $n_1, n_2, \dots$?
— The degree of the polynomial $f(x)$ is n,

$$f(x) = A(x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\dots(x-\alpha_r)^{m_r}$$

$$n = m_1 + m_2 + \dots + m_r$$

Example:

$$\frac{1}{r^2 - 2ar + 1} = \frac{1}{h(x)}$$

$$h(\alpha) = 0 \Rightarrow \alpha = a \pm \sqrt{a^2-1}$$

$$\frac{1}{h(x)} = \frac{1}{(x-\alpha-\sqrt{a^2-1})(x-\alpha+\sqrt{\alpha^2-1})} = \frac{A_1}{x-\alpha-\sqrt{\alpha^2-1}} + \frac{A_2}{x-\alpha+\sqrt{\alpha^2-1}}$$