A modification of the Tsai–Wu failure criterion for the biaxial strength of paper

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ABSTRACT: Prediction of the in-plane biaxial strength of paper by the Tsai–Wu failure criterion has in general proved to be in good agreement with experimental data. However, the Tsai–Wu criterion overestimates the strength in some parts of the biaxial tension region. This shortcoming can be corrected by adding a pair of cubic terms to the existing Tsai–Wu criterion. The modified Tsai–Wu criterion presented here predicts the biaxial strength of paper more accurately than the Tsai–Wu criterion. The extra pair of parameters introduced is determined from data obtained by standard tests.

KEYWORDS: Bag papers, burst strength, criteria, equations, failure, kraft papers, liner boards, parameters, shear stress, tensile strength, theories.

The strength of paper under biaxial loading is of great importance in many industrial applications. Since the early days of industrial use of paper, the burst test *(1)* has been used to characterize, in an uncontrolled way, the biaxial strength of paper. Controlled experiments for testing the biaxial tensile strength of paper have been reported in the literature *(2–4)*. However, efforts to fully characterize the biaxial strength of paper for every stress state have proved to be both tedious and difficult.

These difficulties have led to the development of theoretical approaches that phenomenologically describe the biaxial failure envelope of paper in the entire stress space, with parameters that are easily measured with standard tests. Several biaxial failure envelopes have been developed, but the Tsai–Wu failure criterion envelope *(5)* has proved to be very accurate in describing the biaxial failure of paper *(3, 4)*. However, the Tsai–Wu criterion overestimates the experimental data in some parts of the biaxial tensile strength envelope. Hill *(6)* showed that this shortcoming can be corrected by adding a pair of cubic terms to a failure criterion. This extra pair of parameters is easily determined by standard tests.

The aim of the present work is to

- Define the modified Tsai–Wu criterion, referred to here as the Hill–Tsai–Wu failure criterion
- Show how to numerically determine the extra pair of parameters for the Hill–Tsai–Wu criterion.

The Hill–Tsai–Wu failure criterion is fitted to experimental biaxial-strength data reported in the literature *(3, 4)*. It is shown that the presented criterion provides a better description of the biaxial failure of paper than the Tsai–Wu criterion.

The Hill–Tsai–Wu failure criterion

The material behavior of paper is assumed to be orthotropic, and the in-plane stresses are preferably related to the principal axes of the material, i.e., the 1-axis coincides with the machine direction (MD), and the 2-axis coincides with the cross-machine direction (CD).

The proposed Hill–Tsai–Wu failure criterion is written for plane stress as

$$\phi = \{(p+q) - [(p\sigma_1+q\sigma_2)/P]\}\,(\sigma_1\sigma_2/T_1T_2) + (F_{11}\sigma_1^2 + 2F_{12}\sigma_1\sigma_2 + F_{22}\sigma_2^2 + F_{66}\sigma_6^2 + F_1\sigma_1 + F_2\sigma_2) = 1 \quad (1)$$

where

P = equibiaxial tensile strength

σ_1 = MD stress

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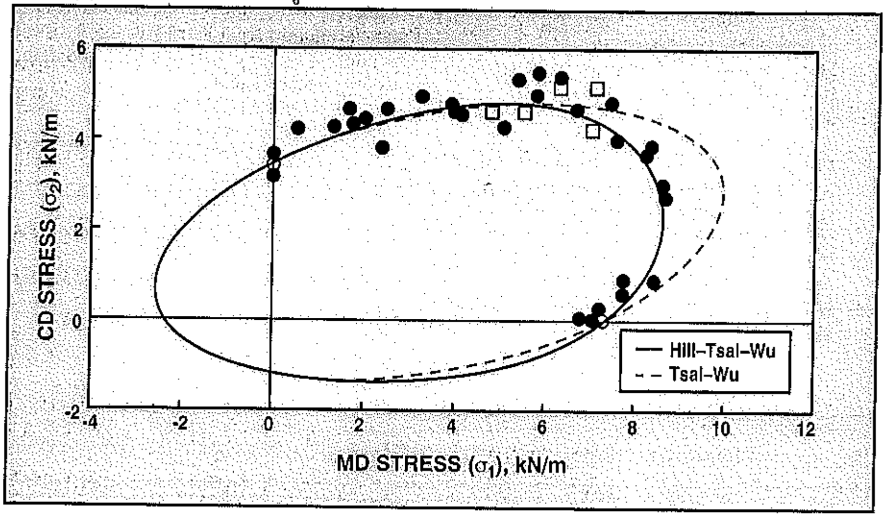
I. Experimental data used to check the accuracy of the proposed Hill–Tsai–Wu failure criterion

	Sack[a]	Linerboard[b]
Basis weight, g/m²	80	125
F_{11}	0.057	0.015
F_{12}	−0.040	−0.013
F_{22}	0.240	0.097
F_1	−0.280	0.150
F_2	−0.550	−0.213
T_1, kN/m	7.3	14.8
T_2, kN/m	3.5	4.5
P, kN/m	4.8	6.2
$d\sigma_1/d\sigma_2$ at $(T_1,0)$	1.340	1.490
$d\sigma_2/d\sigma_1$ at $(0,T_2)$	0.533	0.426

[a]Data from de Ruvo *et al.* *(3)*
[b]Data from Fellers *et al.* *(4)*

1. Plots of Hill–Tsai–Wu and Tsai–Wu failure criteria based on biaxial-strength data for sack paper *(3)*. The shear stress $\sigma_6 = 0$.



σ_2 = CD stress

σ_6 = shear stress

F_{11} = $1/T_1C_1$

F_{12} = $(1/2P^2)(1-\{P[(1/T_1)-(1/C_1)+(1/T_2)-(1/C_2)]\} - \{P^2[(1/T_1C_1)+(1/T_2C_2)]\})$

F_{22} = $1/T_2C_2$

F_{66} = $1/T_{12}C_{12}$

F_1 = $(1/T_1) - (1/C_1)$

F_2 = $(1/T_2) - (1/C_2)$

T_1 = MD tensile strength

T_2 = CD tensile strength

C_1 = MD compressive strength

C_2 = CD compressive strength

T_{12} = tensile shear strength

C_{12} = compressive shear strength

The nondimensional parameters p and q are found from a differentiation of Eq. 1 with respect to σ_2 applied at $(\sigma_1,\sigma_2) = (T_1,0)$ and σ_1 applied at $(\sigma_1,\sigma_2) = (0,T_2)$, where the criteria are satisfied identically. The obtained equation system of p and q has the following solution.

$$[(1/T_1)+(1/T_2)-(1/P)]p = \{(P-T_2)[(F_1/T_1)+2F_{11}](d\sigma_1/d\sigma_2)\} - P\{[(F_2/T_2)+2F_{22}](d\sigma_2/d\sigma_1)\} + P[(F_2/T_1)-(F_1/T_2)] - T_2[(F_2/T_1)+2F_{12}] \quad (2)$$

$$[(1/T_1)+(1/T_2)-(1/P)]q = \{(P-T_1)[(F_2/T_2)+2F_{22}](d\sigma_2/d\sigma_1)\} - P\{[(F_1/T_1)+2F_{11}](d\sigma_1/d\sigma_2)\} + P[(F_1/T_2)-(F_2/T_1)] - T_1[(F_1/T_2)+2F_{12}] \quad (3)$$

where $d\sigma_1/d\sigma_2$ is found from Eq. 1 *(6)* by means of the normality flow-rule, assuming the dominance of plastic strains at the failure,

$$-d\sigma_2/d\sigma_1 = (\partial\phi/\partial\sigma_1)/(\partial\phi/\partial\sigma_2) = d\varepsilon_1/d\varepsilon_2 \quad (4)$$

and the contour of the Hill–Tsai–Wu failure criterion is

$$\phi(\sigma_1,\sigma_2,\sigma_6) = \text{constant}$$

So, under uniaxial tension we have

$$d\sigma_1/d\sigma_2 = -d\varepsilon_2/d\varepsilon_1 = n_1 \text{ at } (T_1,0) \quad (5)$$

$$d\sigma_2/d\sigma_1 = -d\varepsilon_1/d\varepsilon_2 = n_2 \text{ at } (0,T_2) \quad (6)$$

where

n_1 = ratio of the transverse to the longitudinal incremental strains under MD loading

n_2 = ratio of the longitudinal to the transverse strains under CD loading

The ratios n_1 and n_2 can be measured straightforwardly by strain gauges. The values are obtained from data over a small range of strain needed to establish their values with reliability.

Experiments

The Hill–Tsai–Wu failure envelope (Eq. 1) was evaluated using experimental data reported in the literature *(3, 4)*. The modeled results were compared with experimental values of biaxial strength for paper reported in those same works.

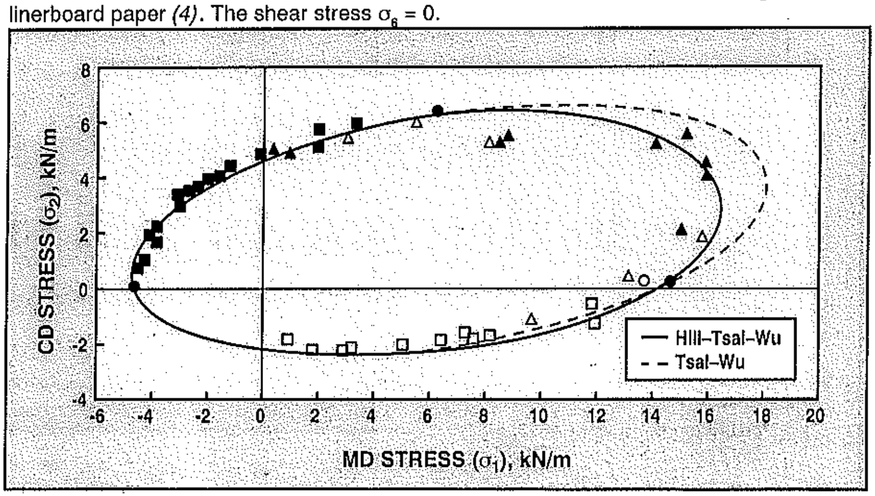
Table I shows the components of the Hill–Tsai–Wu criterion estimated from the biaxial strength data reported in the literature *(3, 4)*. Insertion of the tabulated components in Table I into Eqs. 1–3 gives the Hill–Tsai–Wu failure envelope for the two sets of experimental biaxial strength data.

In **Figs. 1 and 2**, with zero shear stress ($\sigma_6 = 0$), it is seen that the Hill–Tsai–Wu criterion much more accurately represents the experimental biaxial failure envelope for paper than the Tsai–Wu criterion, where $p = 0$ and $q = 0$ in Eq. 1.

Summary

The main purpose of the present work was to suggest a failure criterion—the Hill–Tsai–Wu failure criterion—that does not overestimate the experimental biaxial tension envelope in the way the Tsai–Wu criterion does. This shortcoming was eliminated by adding a specific polynomial of third degree *(6)* to the Tsai–Wu failure criterion. The only negative aspect is that two nondimensional parameters, p and q, are introduced into the proposed Hill–Tsai–Wu failure criterion for paper. However, the parameters p and q are explicitly obtained from the longitudinal and transverse strength in tension and compression plus the equibiaxial tension strength and the ratios of the longitudinal-to-transverse increment strains in the MD and CD loading directions, as seen in Eqs. 2 and 3. ◼

2. Plots of Hill–Tsai–Wu and Tsai–Wu failure criteria based on biaxial-strength data for linerboard paper *(4)*. The shear stress $\sigma_6 = 0$.



Literature cited

1. TAPPI T 403 om-85, "Bursting strength of paper."
2. Setterholm, V. C., Benson, R., and Kuenzi, E. W., *Tappi* 51(5): 196(1968).
3. de Ruvo, A., Carlsson, L., and Fellers, C., *Tappi* 63(5): 133(1980).
4. Fellers, C., Westerlind, B., and de Ruvo, A., *7th Fundamental Research Symposium Proceedings*, Cambridge, U.K., 1981, p. 527.
5. Tsai, S. W. and Wu, E. M., *J. Composite Matl.* 5(1): 58(1971).
6. Hill, R., *Intern. J. Mech. Sci.* 35(1): 19(1993).

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