Some materials science (3 cpu) *LM00EE87*

Hiukan materiaalitiedettä

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Lectures 18 h, exercises 28 h, literature and examination 28 h

I Tensorial elasticity

Boundary displacement
Original length
Engineering strain

True strain, infininitesimal strain

Total differential of displacement, Cauchy strain tensor
Symmetry

Force
Gross-sectional area
Engineering (nominal) stress

Total differential of force, Cauchy stress tensor
Symmetry

Total differential of strain, Compliance
Total differential of stress, Stiffness

Symmetry of Compliance and Stiffness
Orthotropic material symmetry
Co-ordinate transformations of Strain, Stress, Complicance and Stiffness

II Newtonian Flow

Internal friction of fluid
Newtonian viscosity
Darcy's Law
Fick's Law and Divergence Theorem
Hagen-Poiseulle law

III Diffusion

Divergence Theorem
Diffusion Equation
Continuity Equation
Thermal Flux Equation
Thermal Conductivity Equation

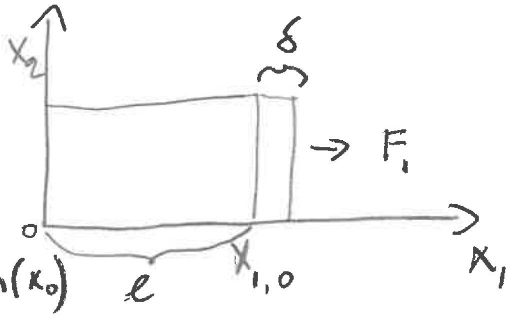
Scaling use of the Diffusion Equation
Solving the Diffusion Equation

Tensorial elasticity

Boundary displacement δ, displacement u
Original length ℓ, $X_{i,0}$
Engineering strain

$$e_i = \frac{\delta}{\ell} = \frac{\delta}{X_{i,0}}$$

Large strain (true large strain)



$$\varepsilon = \int_{x_0}^{x_0+u} \frac{dx}{x} = \Big|_{x_0}^{x_0+u} \ln(x) = \ln(x_0+u) - \ln(x_0)$$

$$= \ln\frac{x_0+u}{x_0} = \ln\left(1+\frac{u}{x_0}\right)$$ True Strain

$\llcorner \ln(1+e)$

$e = \frac{u}{x_0}$ Engineering strain

True strain, infininitesimal strain, local strain

$$\varepsilon_{ij} = \lim_{\Delta X_j \to 0} \frac{\Delta u_i}{\Delta X_j} = \frac{\partial u_i}{\partial X_j}$$

$$\delta_i = u_i = \int_0^{X_{j,0}} \frac{\partial u_i}{\partial X_j}\, dX_j = \int_0^{X_{j,0}} \varepsilon_{ij}\, dX_j$$

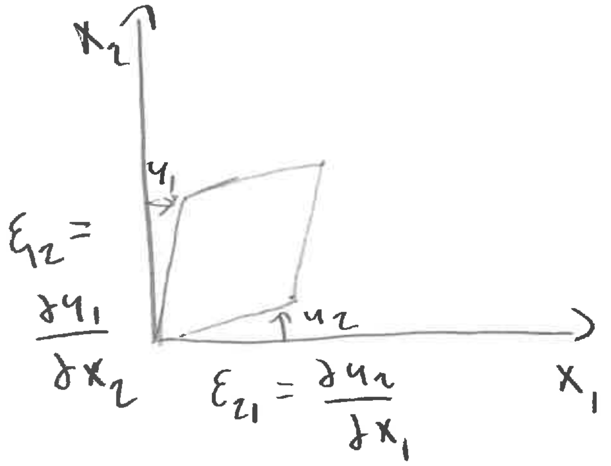
Total differential of displacement

➢ Cauchy strain tensor

$$du_i = \frac{\partial u_i}{\partial x_j} dx_j = \frac{\partial u_i}{\partial x_1} dx_1 + \frac{\partial u_i}{\partial x_2} dx_2 + \frac{\partial u_i}{\partial x_3} dx_3$$

$$du_i = \varepsilon_{ij} dx_j$$

$$\begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$

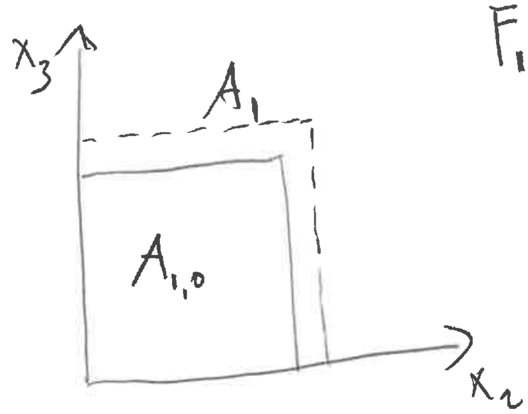


Why is the strain tensor symmetric?

Force
Gross-sectional area
Engineering (nominal) stress

$$\sigma = \frac{F}{A_\perp}$$

$$F_1 = F_{1,0} + \frac{\partial F_1}{\partial A_1} dA_1 = F_{1,0} + \sigma_{11}\, dA_1$$
$$= F_{1,0} + \sigma_{11}(A_1 - A_{1,0})$$



Total differential of Force:

$$dF_i = \frac{\partial F_i}{\partial A_j} dA_j = \frac{\partial F_i}{\partial A_1} dA_1 + \frac{\partial F_i}{\partial A_2} dA_2 + \frac{\partial F_i}{\partial A_3} dA_3$$

$$dF_i = \sigma_{ij}\, dA_j$$

$$\begin{pmatrix} dF_1 \\ dF_2 \\ dF_3 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} dA_1 \\ dA_2 \\ dA_3 \end{pmatrix}$$

Why symmetric?

Total differential of strain

➔ Compliance

$$d\varepsilon_{ij} = \frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}} d\sigma_{kl}$$

$$d\varepsilon_{ij} = S_{ijkl}\, d\sigma_{kl}$$

$$\varepsilon_{ij} = S_{ijkl}\, \sigma_{kl}$$

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{32} \end{pmatrix} = \begin{matrix} S_{1111} & S_{1122} & S_{1133} & S_{1112} & S_{1113} & S_{1121} & S_{1123} & S_{1131} & S_{1132} \\ S_{2211} & S_{2222} & S_{2233} & S_{2212} & S_{2213} & S_{2221} & S_{2223} & S_{2231} & S_{2232} \\ S_{3311} & S_{3322} & S_{3333} & S_{3312} & S_{3313} & S_{3321} & S_{3323} & S_{3331} & S_{3332} \\ S_{1211} & S_{1222} & S_{1233} & S_{1212} & S_{1213} & S_{1221} & S_{1223} & S_{1231} & S_{1232} \\ S_{1311} & S_{1322} & S_{1333} & S_{1312} & S_{1313} & S_{1321} & S_{1323} & S_{1331} & S_{1332} \\ S_{2111} & S_{2122} & S_{2133} & S_{2112} & S_{2113} & S_{2121} & S_{2123} & S_{2131} & S_{2132} \\ S_{2311} & S_{2322} & S_{2333} & S_{2312} & S_{2313} & S_{2321} & S_{2323} & S_{2331} & S_{2332} \\ S_{3111} & S_{3122} & S_{3133} & S_{3112} & S_{3113} & S_{3121} & S_{3123} & S_{3131} & S_{3132} \\ S_{3211} & S_{3222} & S_{3233} & S_{3212} & S_{3213} & S_{3221} & S_{3223} & S_{3231} & S_{3232} \end{matrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{21} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{32} \end{pmatrix}$$

Total differential of stress

- Stiffness

$$d\sigma_{ij} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} d\varepsilon_{kl}$$

$$d\sigma_{ij} = Q_{ijkl}\, d\varepsilon_{kl}$$

$$\sigma_{ij} = Q_{ijkl}\, \varepsilon_{kl}$$

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{21} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{32} \end{pmatrix} = \begin{matrix} Q_{1111} & Q_{1122} & Q_{1133} & Q_{1112} & Q_{1113} & Q_{1121} & Q_{1123} & Q_{1131} & Q_{1132} \\ Q_{2211} & Q_{2222} & Q_{2233} & Q_{2212} & Q_{2213} & Q_{2221} & Q_{2223} & Q_{2231} & Q_{2232} \\ Q_{3311} & Q_{3322} & Q_{3333} & Q_{3312} & Q_{3313} & Q_{3321} & Q_{3323} & Q_{3331} & Q_{3332} \\ Q_{1211} & Q_{1222} & Q_{1233} & Q_{1212} & Q_{1213} & Q_{1221} & Q_{1223} & Q_{1231} & Q_{1232} \\ Q_{1311} & Q_{1322} & Q_{1333} & Q_{1312} & Q_{1313} & Q_{1321} & Q_{1323} & Q_{1331} & Q_{1332} \\ Q_{2111} & Q_{2122} & Q_{2133} & Q_{2112} & Q_{2113} & Q_{2121} & Q_{2123} & Q_{2131} & Q_{2132} \\ Q_{2311} & Q_{2322} & Q_{2333} & Q_{2312} & Q_{2313} & Q_{2321} & Q_{2323} & Q_{2331} & Q_{2332} \\ Q_{3111} & Q_{3122} & Q_{3133} & Q_{3112} & Q_{3113} & Q_{3121} & Q_{3123} & Q_{3131} & Q_{3132} \\ Q_{3211} & Q_{3222} & Q_{3233} & Q_{3212} & Q_{3213} & Q_{3221} & Q_{3223} & Q_{3231} & Q_{3232} \end{matrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{32} \end{pmatrix}$$

Now, stress and strain tensors are symmetric -> six independent components.

Is it possible to reduce the 9*9 matrices into 6*6?

Indeed…

$$\begin{array}{c} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\,\varepsilon_{12} \\ 2\,\varepsilon_{13} \\ 2\,\varepsilon_{23} \end{array} \qquad \begin{array}{cccccc} S_{1111} & S_{1122} & S_{1133} & 2\,S_{1112} & 2\,S_{1113} & 2\,S_{1123} \\ S_{2211} & S_{2222} & S_{2233} & 2\,S_{2212} & 2\,S_{2213} & 2\,S_{2223} \\ S_{3311} & S_{3322} & S_{3333} & 2\,S_{3312} & 2\,S_{3313} & 2\,S_{3323} \\ 2\,S_{1211} & 2\,S_{1222} & 2\,S_{1233} & 4\,S_{1212} & 4\,S_{1213} & 4\,S_{1223} \\ 2\,S_{1311} & 2\,S_{1322} & 2\,S_{1333} & 4\,S_{1312} & 4\,S_{1313} & 4\,S_{1323} \\ 2\,S_{2311} & 2\,S_{2322} & 2\,S_{2333} & 4\,S_{2312} & 4\,S_{2313} & 4\,S_{2323} \end{array} \qquad \begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{array}$$

$$\begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{array} \qquad \begin{array}{cccccc} Q_{1111} & Q_{1122} & Q_{1133} & Q_{1112} & Q_{1113} & Q_{1123} \\ Q_{2211} & Q_{2222} & Q_{2233} & Q_{2212} & Q_{2213} & Q_{2223} \\ Q_{3311} & Q_{3322} & Q_{3333} & Q_{3312} & Q_{3313} & Q_{3323} \\ Q_{1211} & Q_{1222} & Q_{1233} & Q_{1212} & Q_{1213} & Q_{1223} \\ Q_{1311} & Q_{1322} & Q_{1333} & Q_{1312} & Q_{1313} & Q_{1323} \\ Q_{2311} & Q_{2322} & Q_{2333} & Q_{2312} & Q_{2313} & Q_{2323} \end{array} \qquad \begin{array}{c} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\,\varepsilon_{12} \\ 2\,\varepsilon_{13} \\ 2\,\varepsilon_{23} \end{array}$$

Symmetry of Compliance and Stiffness

Strain energy density $\hat{w} = \frac{1}{2} \varepsilon_{ij} \sigma_{ij}$

$$\frac{\partial \hat{w}}{\partial \varepsilon_{ij}} = \sigma_{ij} \qquad \frac{\partial \hat{w}}{\partial \sigma_{ij}} = \varepsilon_{ij}$$

on the other hand

$$\hat{w} = \frac{1}{2}\left(S_{ijkl}\sigma_{kl}\right)\sigma_{ij}$$

$$\frac{\partial \hat{w}}{\partial \sigma_{mn}} = \frac{1}{2} S_{mnkl}\sigma_{kl} + \frac{1}{2} S_{ijmn}\sigma_{ij}$$

$$= \frac{1}{2}\left(S_{mnij} + S_{ijmn}\right)\sigma_{ij}$$

but $\varepsilon_{mn} = S_{mnij}\sigma_{ij}$

$$\frac{1}{2}\left(S_{mnij} + S_{ijmn}\right) = S_{mnij}$$

$$\Rightarrow S_{mnij} = S_{ijmn}$$

$$\boxed{\begin{aligned}\tfrac{1}{2}(a+b) &= a\\ \tfrac{1}{2}b &= \tfrac{1}{2}a\end{aligned}}$$

Similarly for stiffness:

$$\hat{w} = \frac{1}{2}\varepsilon_{ij}\left(Q_{ijkl}\varepsilon_{kl}\right)$$

$$\frac{\partial \hat{w}}{\partial \varepsilon_{mn}} = \frac{1}{2} Q_{mnkl}\varepsilon_{kl} + \frac{1}{2} Q_{ijmn}\varepsilon_{ij} = \frac{1}{2}\left(Q_{mnij} + Q_{ijmn}\right)\varepsilon_{mn}$$

$$= \sigma_{mn} = Q_{mnij}\varepsilon_{ij}$$

$\Rightarrow \quad Q_{mnij} = Q_{ijmn} \quad \Rightarrow$ 45 independent components

Further symmetries of Compliance and Stiffness

Due to symmetry of the strain tensor:

$$Q_{ijkl} = Q_{ijlk}$$
$$S_{ijkl} = S_{jikl}$$

Due to symmetry of the stress tensor:

$$Q_{ijkl} = Q_{jikl}$$
$$S_{ijkl} = S_{ijlk}$$

Combining both symmetries:

$$Q_{ijkl} = Q_{jilk}$$
$$S_{ijkl} = S_{ijlk}$$

⇨ Matrices can be written as 6*6 square matrices

$\sigma_{ij} = \sigma_{ji}$

Engineering shear strain $e_{ij} = \varepsilon_{ij} + \varepsilon_{ji} = 2\varepsilon_{ij}$

Symmetry -> 21 independent components

Orthotropic material symmetry

Orthotropic Material, on-axis co-ordinates

- 2 perpendicular lines of reflection symmetry (2d)
- 3 perpendicular planes of reflection symmetry (3d)

- Shear stresses are not linked to normal strains
- Normal stresses are not linked to shear strains

- Shear strains are not linked to normal stresses
- Normal strains are not linked to shear stresses

Why are shear stresses not linked to normal strains, etc?

Any reflections wrt a symmetry plane changes the sign of one co-ordinate (or displacement)

=> Normal strains and stresses invariant in symmetry transformations

=> Shear strains and stresses change sign in symmetry transformations

=> Stiffnesses and compliances linking shear and normal strains/stresses should change sign in symmetry transformations. The only constant being equal to its negative is zero – this of course applies also to material constants!
rthotropic Material, continued

Why are shear stresses σ_{ij} not linked to shear strains σ_{jk}?

Reflection wrt symmetry plane (j,k) changes the sign of the co-ordinate (or displacement) i. Thus σ_{ij} changes sign, but σ_{jk} does not. The only material constant equal to its negative is again zero...

Compliance Matrix – 9 independent components

$$
\begin{array}{c|ccccccccc|c}
\varepsilon_{11} & S_{1111} & S_{1122} & S_{1133} & & & & & & & \sigma_{11} \\
\varepsilon_{22} & S_{2211} & S_{2222} & S_{2233} & & & & & & & \sigma_{22} \\
\varepsilon_{33} & S_{3311} & S_{3322} & S_{3333} & & & & & & & \sigma_{33} \\
 & & & & S_{1212} & & S_{1221} & & & & \\
 & & & & & S_{1313} & & & S_{1331} & & \\
 & & & & S_{2112} & & S_{2121} & & & & \\
 & & & & & & & S_{2323} & & S_{2332} & \\
 & & & & & S_{3113} & & & S_{3131} & & \\
 & & & & & & & S_{3223} & & S_{3232} &
\end{array}
$$

Stiffness Matrix – 9 independent components

$$
\begin{array}{c|ccccccccc|c}
\sigma_{11} & Q_{1111} & Q_{1122} & Q_{1133} & & & & & & & \varepsilon_{11} \\
\sigma_{22} & Q_{2211} & Q_{2222} & Q_{2233} & & & & & & & \varepsilon_{22} \\
\sigma_{33} & Q_{3311} & Q_{3322} & Q_{3333} & & & & & & & \varepsilon_{33} \\
 & & & & Q_{1212} & & Q_{1221} & & & & \\
 & & & & & Q_{1313} & & & Q_{1331} & & \\
 & & & & Q_{2112} & & Q_{2121} & & & & \\
 & & & & & & & Q_{2323} & & Q_{2332} & \\
 & & & & & Q_{3113} & & & Q_{3131} & & \\
 & & & & & & & Q_{3223} & & Q_{3232} &
\end{array}
$$

Co-ordinate transformations of Strain, Stress, Complicance and Stiffness

Off-axis mechanical behavior of orthotropic materials

Off-axis stresses known – determine off-axis strains
1/ Stress transformation to on-axis $\overline{\sigma} = T\,\overline{\sigma}'$
2/ Use on-axis compliance matrix to determine on-axis strains $\overline{\varepsilon} = S\,\overline{\sigma}$
3/ Negative transformation of strain to off-axis $\overline{\varepsilon}' = K^{-1}\,\overline{\varepsilon}$

Off-axis strains known – determine off-axis stresses
1/ Strain transformation to on-axis ~~$\overline{\varepsilon}' = K\,\overline{\varepsilon}$~~ $\overline{\varepsilon} = K\,\overline{\varepsilon}'$
2/ Use on-axis stiffness matrix to determine on-axis stresses $\overline{\sigma} = Q\,\overline{\varepsilon}$
3/ Negative transformation of stress to off-axis $\overline{\sigma}' = T^{-1}\,\overline{\sigma}$

This cannot be done using the Engineering constants!

Co-ordinate transformation of Compliance

1/ Transform stress to on-axis $\overline{\sigma} = T\,\overline{\sigma'}$

2/ Use on-axis compliance to compute strain $\overline{\varepsilon} = S\,\overline{\sigma}$

3/ Transform strain to off-axis $\overline{\varepsilon'} = K^{-1}\,\overline{\varepsilon}$

4/ Identify off-axis Compliance components – they are given as polynomials of on-axis compliance components! $\overline{\varepsilon'} = K^{-1} S\,T\,\overline{\sigma'} = S'\,\overline{\sigma'}$

Co-ordinate transformation of Stiffness

1/ Transform strain to on-axis ~~$\overline{\varepsilon'} = K\,\overline{\varepsilon}$~~ $\overline{\varepsilon} = K\varepsilon'$

2/ Use on-axis stiffness to compute stress $\overline{\sigma} = Q\,\overline{\varepsilon}$

3/ Transform stress to off-axis $\overline{\sigma'} = T^{-1}\,\overline{\sigma}$

4/ Identify off-axis Stiffness components – they are given as polynomials of on-axis stiffness components! $\overline{\sigma'} = T^{-1} Q\,K\,\overline{\varepsilon'} = Q'\,\overline{\varepsilon'}$

How do we determine Compliance Matrix experimentally?

Once we know Compliance Matrix, how do we get to know Stiffness Matrix?

If we know Stiffness Matrix, how do we get to know Compliance Matrix?

$$\begin{array}{ccccccccc}
S_{1111} & S_{1122} & S_{1133} & & & & & & \\
S_{2211} & S_{2222} & S_{2233} & & & & & & \\
S_{3311} & S_{3322} & S_{3333} & & & & & & \\
 & & & S_{1212} & & S_{1221} & & & \\
 & & & & S_{1313} & & & S_{1331} & \\
 & & & S_{2112} & & S_{2121} & & & \\
 & & & & & & S_{2323} & & S_{2332} \\
 & & & & S_{3113} & & & S_{3131} & \\
 & & & & & & S_{3223} & & S_{3232}
\end{array}$$

$$\begin{array}{ccccccccc}
Q_{1111} & Q_{1122} & Q_{1133} & & & & & & \\
Q_{2211} & Q_{2222} & Q_{2233} & & & & & & \\
Q_{3311} & Q_{3322} & Q_{3333} & & & & & & \\
 & & & Q_{1212} & & Q_{1221} & & & \\
 & & & & Q_{1313} & & & Q_{1331} & \\
 & & & Q_{2112} & & Q_{2121} & & & \\
 & & & & & & Q_{2323} & & Q_{2332} \\
 & & & & Q_{3113} & & & Q_{3131} & \\
 & & & & & & Q_{3223} & & Q_{3232}
\end{array}$$

How do we invert a matrix?

Gaussian Elimination
Cramer's Rule (cofactors)
Cholensky Decomposition
Eigendecomposition
Cayley-Hamilton

II Newtonian flow

Newtonian Viscosity

Interaction of molecules -> internal fluid friction

$$\bar{\sigma} = Q\,\bar{\varepsilon}$$

$$\to \sigma_{ij} = \mu\,\dot{\varepsilon}_{ij} = \mu\,\frac{\partial \varepsilon_{ij}}{\partial t} = \mu\,\frac{\partial^2 u_i}{\partial t \partial x_j}$$

Gravity Drainage Experiment

$$dV_1 = -dV_2$$
$$\Delta p = p_2 - p_1$$

Darcy's Law

$$\frac{dV}{dt} \propto \Delta p$$
$$\frac{dV_1}{dt} = \frac{\Delta p\ A}{\mu\ R}$$
$$\frac{dV_2}{dt} = -\frac{\Delta p\ A}{\mu\ R} = -\frac{\Delta p\ A}{\mu\ b\ SFR}$$
$$\frac{dV_2}{A\ dt} = -\frac{\Delta p}{\mu\ b\ SFR} \to -\frac{K}{\mu}\frac{dp}{dx}$$

Flux Equation

1-d Fick's law

$$\frac{\partial^2 \Omega}{\partial t \partial A} = -D\frac{\partial^2 \Omega}{\partial x \partial V}$$

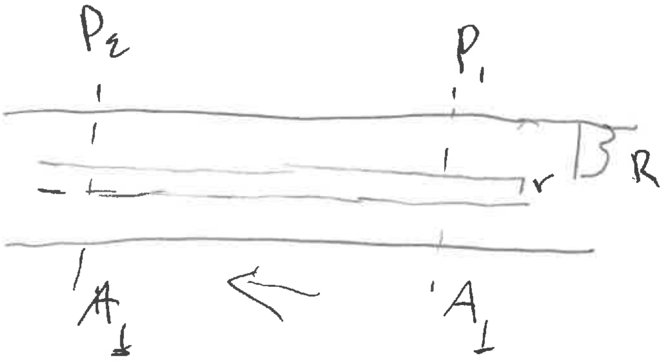
3-d Fick's law

$$\bar{J} = -[D]\nabla\theta$$

or

$$J_i = \frac{\partial^2 \Omega}{\partial t \partial A_i} = -D_{ij}\frac{\partial^2 \Omega}{\partial x_j \partial V}$$

Hagen-Poiseulle Law



$$-(p_2 - p_1)\pi r^2 + \sigma_{lr} 2\pi r\,\Delta l = 0$$

$$-(p_2 - p_1)\pi r^2 + \mu\,\dot{\varepsilon}_{lr}\, 2\pi r\,\Delta l = 0$$

$$\dot{\varepsilon}_{lr} = \frac{d}{dt}\frac{du}{dr} = \frac{d}{dr}\frac{du}{dt} = \frac{d}{dr}v$$

$$\dot{\varepsilon}_{lr} = \frac{(p_2 - p_1)r}{2\mu\,\Delta l} \approx \frac{dp}{dl}\frac{r}{2\mu}$$

$$\frac{dv}{dr} = \frac{dp}{dl}\frac{r}{2\mu}$$

$$dv = \frac{dp}{dl}\frac{r}{2\mu}dr$$

$$v = \frac{dp}{dl}\frac{r^2}{4\mu}\cancel{dr} + C$$

$$v(r = R) = 0 \;->\; C = -\frac{dp}{dl}\frac{R^2}{4\mu}$$

$$v(r) = \frac{dp}{dl}\frac{r^2 - R^2}{4\mu}$$

$$\frac{dV}{dt} = \int_0^R 2\pi r\, v(r)\; dr = -\frac{dp}{dl}\frac{\pi}{2\mu}\int_0^R (r\,R^2 - r^3)\; dr = -\frac{dp}{dl}\frac{\pi}{2\mu}\int_0^R (r\,R^2 - r^3)\; dr = -\frac{dp}{dl}\frac{\pi R^4}{8\mu}$$

$$\Big/_0^R \frac{r^2 R^2}{2} - \frac{r^4}{4}$$

Flux

$$\frac{dV}{A_\perp\, dt} = -\frac{dp}{\pi R^2 dl}\frac{\pi R^4}{8\mu} = -\frac{dp}{dl}\frac{R^2}{8\mu}$$

$$\frac{R^4}{2} - \frac{R^4}{4} = \frac{R^4}{4}$$

Poros System Consisting of Parallel Tubes

$$\frac{dV}{dt}=\left(\frac{dV}{dt}\right)_1+\left(\frac{dV}{dt}\right)_2+\ldots+\left(\frac{dV}{dt}\right)_n$$

$$\frac{dV}{A_\perp dt}=\frac{1}{A_\perp}\left[\left(\frac{dV}{dt}\right)_1+\left(\frac{dV}{dt}\right)_2+\ldots+\left(\frac{dV}{dt}\right)_n\right]$$

$$\frac{dV}{A_\perp dt}=\frac{1}{A_\perp}\left[A_1\left(\frac{dV}{A\,dt}\right)_1+A_2\left(\frac{dV}{A\,dt}\right)_2+\ldots+A_n\left(\frac{dV}{A\,dt}\right)_n\right]$$

If all tubes are of the same size

$$\frac{dV}{A_\perp dt}=\frac{n\,A_1}{A_\perp}\left(\frac{dV}{A\,dt}\right)_1=\frac{n\,A_1}{A_\perp}\left(-\frac{dp}{dl}\frac{R^2}{8\mu}\right)=\mathrm{P}\left(-\frac{dp}{dl}\frac{R^2}{8\mu}\right)$$

If all tubes are not of the same size

$$\frac{dV}{A_\perp dt}=\frac{A_1}{A_\perp}\left(-\frac{dp}{dl}\frac{R_1^2}{8\mu}\right)+\frac{A_2}{A_\perp}\left(-\frac{dp}{dl}\frac{R_2^2}{8\mu}\right)+\ldots+\frac{A_n}{A_\perp}\left(-\frac{dp}{dl}\frac{R_n^2}{8\mu}\right)$$

$$=-\frac{1}{A_\perp}\frac{dp}{dl}\frac{1}{8\mu}\left(A_1R_2^2+A_2R_2^2+\ldots+A_nR_n^2\right)$$

$$=-\frac{A_p}{A_\perp}\frac{dp}{dl}\frac{1}{8\mu}\frac{A_1R_2^2+A_2R_2^2+\ldots+A_nR_n^2}{A_p}$$

In the continuum form

$$\frac{dV}{A_\perp dt}=-\mathrm{P}\frac{dp}{dl}\frac{1}{8\mu}\int p(A)\;R^2\;dA$$

$$=-\mathrm{P}\frac{dp}{dl}\frac{1}{8\mu}\int p(A)\;R^2\;2\pi R\;dR$$

$$=-\mathrm{P}\frac{dp}{dl}\frac{1}{8\mu}\int p(R)\;R^2\;dR$$

$$\int p(A)\;dA=1$$

$$\int p(A)\;d(\pi R^2)=1$$

$$\frac{d\;\pi R^2}{d\;R}=2\;\pi R$$

$$d\;\pi R^2=2\pi R\;\;d\;R$$

$$\int p(A)\;2\pi R\;\;d\;R=1$$

$$p(R)=p(A)\;2\pi R$$

Recall Darcy's Law - Flux Equation

$$\frac{dV}{dt} \propto \Delta p$$

$$\frac{d^2V}{dA_\perp dt} = -\frac{\Delta p}{\mu\, b\, SFR} \rightarrow -\frac{K}{\mu}\frac{dp}{dx}$$

Porosity effect

$$\frac{d}{dA_\perp}\frac{dV}{dt} = \frac{dN_P}{dA_\perp}\frac{d}{dN_P}\frac{dV}{dt}$$

$$\mathrm{P} = \frac{dA_p}{dA_\perp} = \frac{dN_p A_1}{dA_\perp} => \frac{dN_p}{dA_\perp} = \frac{\mathrm{P}}{A_1}$$

$$\frac{d}{dN}\frac{dV}{dt} = \left(\frac{dV}{dt}\right)_1 = -\frac{dp}{dl}\frac{\pi R^4}{8\mu}$$

Flux within a porous body

$$\frac{d}{dA_\perp}\frac{dV}{dt} = \frac{dN_P}{dA_\perp}\frac{d}{dN_P}\frac{dV}{dt} = \frac{\mathrm{P}}{\pi R^2}\left(-\frac{dp}{dl}\frac{\pi R^4}{8\mu}\right) = \mathrm{P}\left(-\frac{dp}{dl}\frac{R^2}{8\mu}\right)$$

$$=> K = \mathrm{P}\frac{R^2}{8}$$

Variable size of pores

$$=> K = \frac{\mathrm{P}}{8}\int R^2\, p(R)\, dR$$

$$\bar{a} = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3$$

III Diffusion

$$\nabla = \hat{e}_1 \frac{\partial}{\partial x_1} + \hat{e}_2 \frac{\partial}{\partial x_2} + \hat{e}_3 \frac{\partial}{\partial x_3}$$

Divergence Theorem

$$\int_V \nabla \cdot \bar{a}\, dV = \oint_s \bar{a} \cdot d\bar{s} \qquad \nabla = \hat{e}_i \frac{\partial}{\partial x_i} \qquad \nabla \cdot \bar{a} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}$$

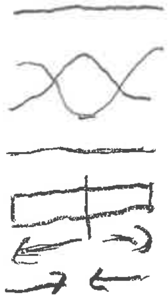
Total flow rate out of volume *V*

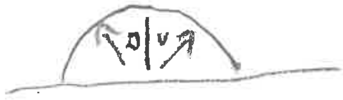
$$-\frac{dQ}{dt} = \oint_s \bar{J} \cdot d\bar{s} = \oint_s -D\, \nabla \phi \cdot d\bar{s}$$

$$-\frac{dQ}{dt} = \oint_s \bar{J} \cdot d\bar{s} = \int_V \nabla \cdot \bar{J}\, dV = \int_V -D\, \nabla^2 \phi\, dV$$

$$Q = \int_V \phi\, dV$$

$$-\frac{dQ}{dt} = -\int_V \frac{\partial \phi}{\partial t}\, dV$$

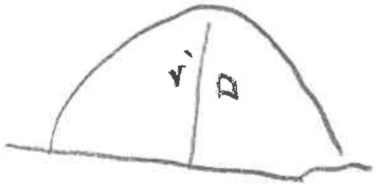




$$\frac{\partial \phi}{\partial t} = D\, \nabla^2 \phi$$ Diffusion Equation

$$\frac{\partial \phi}{\partial t} + \nabla \cdot \bar{J} = 0$$ Continuity Equation

$$\frac{\partial J}{\partial x} = 0 \Rightarrow J = C \qquad \frac{d\, \text{constan}}{dx} = 0$$



$$r' = 7r$$

Thermal Flux Equation

$\bar{J} = -D\, \nabla \phi$ or ~~$\frac{\partial^2 Q}{\partial A_j \partial t} \hat{e}_j = -\delta_{ijk} D_i \hat{e}_j \frac{\partial^2 Q}{\partial x_k \partial V}$~~ → 20b

$$\left[\frac{J}{s\, m^2}\right] = \left[\frac{m^2}{s}\right]\left[\frac{J}{m^4}\right]$$

Thermal Conductivity Equation

$\bar{J} = -D\, \nabla \phi$ or ~~$\frac{\partial^2 Q}{\partial A_j \partial t} \hat{e}_j = -\delta_{ijk} \kappa_i \hat{e}_j \frac{\partial T}{\partial x_k}$~~ → 20b

$$\left[\frac{J}{s\, m^2}\right] = \left[\frac{J}{m\, s\, K}\right]\left[\frac{K}{m}\right]$$

~~$\kappa = D$~~ $\frac{\partial^2 Q}{\partial T \partial V}$ → 20b Thermal conductivity

$\frac{\partial^2 Q}{\partial T \partial V}$ Volumetric heat capacity

III Diffusion

Divergence Theorem

$$\int_V \nabla\cdot\bar{a}\,dV = \oint_s \bar{a}\cdot\,d\bar{s} \qquad \nabla = \hat{e}_i\,\frac{\partial}{\partial x_i} \qquad \nabla\cdot\bar{a} = \frac{\partial a_1}{\partial x_1}+\frac{\partial a_2}{\partial x_2}+\frac{\partial a_3}{\partial x_3}$$

Total flow rate out of volume *V*

$$-\frac{dQ}{dt} = \oint_s \bar{J}\cdot\,d\bar{s} = \oint_s -D\,\nabla\phi\cdot\,d\bar{s}$$

$$-\frac{dQ}{dt} = \oint_s \bar{J}\cdot\,d\bar{s} = \int_V \nabla\cdot\bar{J}\,dV = \int_V -D\,\nabla^2\phi\,dV$$

$$Q = \int_V \phi\,dV$$

$$-\frac{dQ}{dt} = -\int_V \frac{\partial\phi}{\partial t}\,dV$$

$$\frac{\partial\phi}{\partial t} = D\,\nabla^2\phi$$

Diffusion Equation

$$\frac{\partial\phi}{\partial t} + \nabla\cdot\bar{J} = 0$$

Continuity Equation

Thermal Flux Equation

$$\bar{J} = -D\,\nabla\phi \qquad \text{or} \qquad \frac{\partial^2 Q}{\partial A_i\partial t} = -D_{ij}\,\frac{\partial^2 Q}{\partial x_j\partial V}$$

$$\left[\frac{J}{s\,m^2}\right] = \left[\frac{m^2}{s}\right]\left[\frac{J}{m^4}\right]$$

Thermal Conductivity Equation

$$\bar{J} = -D\,\nabla\phi \qquad \text{or} \qquad \frac{\partial^2 Q}{\partial A_i\partial t} = -\kappa_{ij}\,\frac{\partial\,T}{\partial x_j}$$

$$\left[\frac{J}{s\,m^2}\right] = \left[\frac{J}{m\;s\;K}\right]\left[\frac{K}{m}\right]$$

$$\kappa_{ij} = D_{ij}\,\frac{\partial^2 Q}{\partial T\partial V}$$

Thermal conductivity

$$\frac{\partial^2 Q}{\partial T\partial V}$$

Volumetric heat capacity

How to use the Diffusion Equation?
Can we use the Diffusion Equation without solving it?

$$\frac{\partial \phi}{\partial t} = D\,\nabla^2 \phi \qquad \text{Diffusion Equation}$$

$$\frac{\partial \phi}{\partial t} = \sum_{i=1}^{3} D_{ij} \frac{\partial^2 \phi}{\partial x_j^{\,2}}$$

Introduce linear size scaling with scaling factor *qt*

$$\frac{\partial \phi}{\partial q^n t} = \sum_{i=1}^{3} D_{ij} \frac{\partial^2 \phi}{\partial (q x_k)^2} \qquad => n=2$$

An application to drying, impregnation, or heating time:

$$\Delta t = \int_{\phi_o}^{\phi_f} \frac{\partial t}{\partial \phi} d\phi$$

$$\Delta t' = \int_{\phi_o}^{\phi_f} q^2 \frac{\partial t}{\partial \phi} d\phi = q^2 \Delta t$$

How to solve the Diffusion Equation?

Steady-state problems:

$$\frac{\partial \phi}{\partial t} = 0 \qquad \Rightarrow \; 0 = \nabla^2 \phi \quad \text{Laplace Eq.}$$

Transient problems:
Fourier Series Solution

Example:

$$\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2}$$

Wall of thickness L
Heater, Power/area=H

Flux at heat source $J(L,t) = -D\frac{\partial u}{\partial x} = -H$ Inhomogeneous boundary condition

Outside generalized temperature $u(0,t) = 0$

Initial generalized temperature $u(x,0) = 0$

Potential function transformation $u(x,t) = v(x,t) + w(x)$

$$\frac{\partial w}{\partial t} = D\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 v}{\partial x^2}\right)$$

Outside generalized temperature $v(0,t) + w(0) = 0$

Initial generalized temperature $v(x,0) + w(x) = 0$

Flux at heat source $J(L,t) = -D\left[\frac{\partial v(L,t)}{\partial x} + \frac{\partial w(L,t)}{\partial x}\right] = -H \Rightarrow$

$$\left[\frac{\partial v(L,t)}{\partial x} + \frac{\partial w(L,t)}{\partial x}\right] = \frac{H}{D}$$

Set $w = \frac{Hx}{D}$, then

$$\frac{\partial v(L,t)}{\partial x} = 0 \qquad \text{Homogeneous boundary condition}$$

$$\frac{\partial v(x,0)}{\partial x} = -\frac{H}{D} \;\Rightarrow\; v(x,0) = -\frac{Hx}{D}$$

$$v(0,t) = 0$$

Let us try to solve $v(x,t)$ by separation of variables:

$$v(x,t) = X(x)T(t)$$

$$\frac{\partial\left[X(x)T(t)\right]}{\partial t} = D\,\frac{\partial^2\left[X(x)T(t)\right]}{\partial x^2}$$

$$X\,T' = D\,X''\,T$$

$$\frac{X''}{X} = \frac{1}{D}\frac{T'}{T} \equiv -\lambda^2$$

$$X'' = -\lambda^2\,X$$

$$X = a\,e^{i\lambda x} + b\,e^{-i\lambda x} = A\,\cos(\lambda x) + B\,\sin(\lambda x)$$

$$T' = -\lambda^2\,D\,T$$

$$T = c\,e^{-\lambda^2 D\,t}$$

$$v(x,t) = \left[A\,\cos(\lambda x) + B\,\sin(\lambda x)\right]e^{-\lambda^2 D\,t}$$

Boundary conditions:

$$v(0,t) = 0 \;=>\; A = 0$$

$$\frac{\partial v(L,t)}{\partial x} = 0 \;=>\; \lambda = \frac{n\,\pi}{2L},\quad n = 1,3,5,7,\ldots$$

One solution:

$$v(x,t) = B\,\sin(\frac{n\,\pi}{2L}x)\;e^{-\frac{n^2\pi^2 D}{4L^2}t}$$

General solution as superposition

$$v(x,t) = \sum_{n\ odd} B_n\sin(\frac{n\,\pi}{2L}x)\;e^{-\frac{n^2\pi^2 D}{4L^2}t}$$

With boundary condition *t*=0

$$v(x,0) = \sum_{n\ odd} B_n\sin(\frac{n\,\pi}{2L}x) = -\frac{H\,x}{D}$$ Identify Fourier sine series

$$\cdot\sin(\frac{n\,\pi}{2L}x)$$

$$\int_{-L}^{L} dx$$

$$\int_{b}^{b+\pi} \sin^2(x)\ dx = \frac{\pi}{2}$$

$$v(x,0) = \sum_{n\ odd} B_n \sin(\frac{n\,\pi}{2L}x) \ = -\frac{H\ x}{D}$$

$$\frac{2L}{2}B_n = \int_{-L}^{L} -\frac{H\ x}{D}\sin(\frac{n\,\pi}{2L}x)dx$$

$$B_n = -\frac{H}{LD}\int_{-L}^{L} x\sin(\frac{n\,\pi}{2L}x)dx$$

$$= -\frac{H}{D}\frac{4L}{n^2\pi^2}\int_{-\frac{n\pi}{2}}^{\frac{n\pi}{2}} \frac{n\,\pi x}{2L}\sin(\frac{n\,\pi}{2L}x)d\frac{n\,\pi x}{2L}$$

$$= -\frac{H}{D}\frac{8L}{n^2\pi^2}\int_{0}^{\frac{n\pi}{2}} y\sin(y)dy$$

$$= -\frac{H}{D}\frac{8L}{n^2\pi^2}\int_{0}^{\frac{n\pi}{2}} y\sin(y)dy$$

$$= -\frac{H}{D}\frac{8L}{n^2\pi^2}\left\{ {}_{0}^{\frac{n\pi}{2}}\Big| y\left[-\cos(y)\right] - \int_{0}^{\frac{n\pi}{2}} -\cos(y)dy \right\}$$

$$= -\frac{H}{D}\frac{8L}{n^2\pi^2}\ {}_{0}^{\frac{n\pi}{2}}\Big|\sin(y) = -\frac{H}{D}\frac{8L}{\pi^2}\frac{(-1)^{\frac{n-1}{2}}}{n^2}$$

$$v(x,0) = -\frac{H}{D}\frac{8L}{\pi^2}\sum_{n\ odd}\frac{(-1)^{\frac{n-1}{2}}}{n^2}\sin(\frac{n\,\pi}{2L}x)$$

$$u(x,t) = w(x) + v(x,t)$$

$$= \frac{H}{D}x - \frac{H}{D}\frac{8L}{\pi^2}\sum_{n\ odd}\frac{(-1)^{\frac{n-1}{2}}}{n^2}\sin(\frac{n\,\pi}{2L}x)e^{-\frac{n^2\pi^2 D}{4L^2}t}$$

Potential function in terms of reduced variables:

$u' = u\frac{H}{LD}$ Reduced heat density

$x' = \frac{x}{L}$ Reduced position

$t' = \frac{\pi^2 D}{4L^2}t$ Reduced time

$$u'(x',t') = x' - \frac{8}{\pi^2}\sum_{n\ odd}\frac{(-1)^{\frac{n-1}{2}}}{n^2}\sin(\frac{n\ \pi}{2}x')e^{-n^2 t'}$$

Does the solution satisfy the Diffusion Equation $\frac{\partial u}{\partial t} = D\ \frac{\partial^2 u}{\partial x^2}$?

$$w(x) = \frac{Hx}{D}$$

$$\frac{\partial^2 w}{\partial x^2} = 0 \implies \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial v}{\partial x} = -\frac{H}{D}\frac{8L}{\pi^2}\sum_{n\ odd}\frac{n\ \pi}{2L}\frac{(-1)^{\frac{n-1}{2}}}{n^2}\cos(\frac{n\ \pi}{2L}x)e^{-\frac{n^2\pi^2 D}{4L^2}t}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{H}{D}\frac{8L}{\pi^2}\sum_{n\ odd}\frac{n^2\pi^2}{4L^2}\frac{(-1)^{\frac{n-1}{2}}}{n^2}\sin(\frac{n\ \pi}{2L}x)e^{-\frac{n^2\pi^2 D}{4L^2}t}$$

$$= \frac{H}{LD}\,2\sum_{n\ odd}\frac{(-1)^{\frac{n-1}{2}}}{n^2}\sin(\frac{n\ \pi}{2L}x)e^{-\frac{n^2\pi^2 D}{4L^2}t}$$

$$\frac{\partial v}{\partial t} = \frac{H}{D}\frac{8L}{\pi^2}\sum_{n\ odd}\frac{n^2\ \pi^2 D}{4L^2}\frac{(-1)^{\frac{n-1}{2}}}{n^2}\sin(\frac{n\ \pi}{2L}x)e^{-\frac{n^2\pi^2 D}{4L^2}t}$$

$$= \frac{H}{L}\,2\sum_{n\ odd}\frac{(-1)^{\frac{n-1}{2}}}{n^2}\sin(\frac{n\ \pi}{2L}x)e^{-\frac{n^2\pi^2 D}{4L^2}t}$$

$\frac{\partial v}{\partial t} = D\ \frac{\partial^2 v}{\partial x^2}$ *PROVEN*

How does the solution become in other problems?

One-dimensional symmetric drying/impregnation problem

Again:

$$\frac{\partial u}{\partial t} = D\ \frac{\partial^2 u}{\partial x^2}$$
$$u(x,t) = X(x)T(t)$$
$$X\ T' = D\ X''\ T$$
$$\frac{X''}{X} = \frac{1}{D}\frac{T'}{T} \equiv -\lambda^2$$

$$X'' = -\lambda^2\ X$$
$$X = a\ e^{i\lambda x} + b\ e^{-i\lambda x} = A\ \cos(\lambda x) + B\ \sin(\lambda x)$$

$$T' = -\lambda^2\ D\ T$$
$$T = c\ e^{-\lambda^2 D\, t}$$

Boundary condition:

$$\frac{\partial X(0)}{\partial x} = 0 \quad \Rightarrow B=0$$
$$X = A\ \cos(\lambda x)$$

Drying problem:

$$\frac{\partial X}{\partial x} \geq 0 \ \Rightarrow\ X = A\ \cos(\frac{n\pi x}{2b}) \qquad n=1, 3, 5, \ldots$$

$$u(x,t) = \sum_{n\ odd} A_n \cos(\frac{n\ \pi}{2b} x)\ e^{-\frac{n^2\pi^2 D}{4b^2}t}$$

$$u(x,0) = \sum_{n\ odd} A_n \cos(\frac{n\ \pi}{2b} x)$$

$$A_n = \frac{2}{b}\int_0^b u(x,0)\ \cos(\frac{n\ \pi}{2b} x)dx$$

$$= \frac{4}{n\ \pi}\int_0^{\frac{n\ \pi}{2}} u(x,0)\ \cos(\frac{n\ \pi}{2b} x)d\frac{n\ \pi x}{2b}$$

$$= \frac{4\ u(x,0)}{n\ \pi}(-1)^{\frac{n-1}{2}}$$

Set $u(x,0) = 1 \ \Rightarrow\ u(x,t) = \sum_{n\ odd} \frac{4}{n\ \pi}(-1)^{\frac{n-1}{2}} \cos(\frac{n\ \pi}{2b} x)\ e^{-\frac{n^2\pi^2 D}{4b^2}t}$ FINE