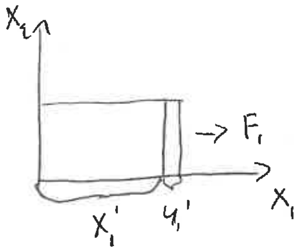
Forest Products Mechanics – Mechanics of Materials! (1)

Normal strain and stress



For small deformations

$$\varepsilon_1 = \frac{u_1'}{X_1'} \quad ? \qquad \boxed{u_1' \ll X_1'}$$

$$\varepsilon_{ij} = \lim_{\Delta x_j \to 0} \frac{\Delta u_i}{\Delta x_j} = \frac{\partial u_i}{\partial x_j}$$

$$\varepsilon_{11} = \lim_{\Delta x_1 \to 0} \frac{\Delta u_1}{\Delta x_1} = \frac{\partial u_1}{\partial x_1}$$

$$u_1' = \int_0^{X_1'} \frac{\partial u_1}{\partial x_1} \, dx_1 = \int_0^{X_1'} \varepsilon_{11} \, dx_1$$

Jos ε_{11} vakio $\quad \varepsilon_{11} X_1'$

$$\Rightarrow \varepsilon_{11} = \frac{u_1'}{X_1'}$$

Large Deformations?

Accumulated Average Strain

Boundary Displacement = relative change in length

$$\varepsilon = \frac{dl}{l} = \frac{dx}{X}$$

$$\bar{\varepsilon} = \int_X^{X+u} \frac{dx}{x} = \Big/_{X}^{X+u} \ln x = \ln(x+u) - \ln x = \ln \frac{x+u}{x} = \ln\left(1 + \frac{u}{x}\right)$$

Logarithmic Strain $\ln\left(1 + \frac{u}{x}\right)$

Engineering Strain $\frac{u}{X}$

Normal Strain Components

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}$$

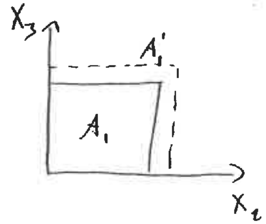
$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2}$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

Stress: Internal Force (2)

At constant strain, displacement depends on length.

Is there some kind of relation between specimen size and Force needed to induce the Strain?



$$F_1' = F_1 + \frac{\partial F_1}{\partial A_1} dA_1 = F_1 + \sigma_{11} \, dA_1$$
$$= F_1 + \sigma_{11} (A_1' - A_1)$$

Normal Stress Components

$$\sigma_{11} = \frac{\partial F_1}{\partial A_1}$$

$$\sigma_{22} = \frac{\partial F_2}{\partial A_2}$$

$$\sigma_{33} = \frac{\partial F_3}{\partial A_3}$$

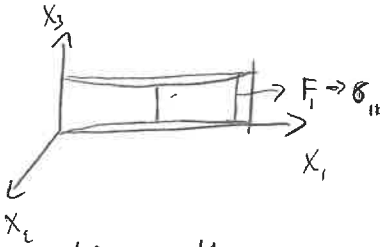
Body: Boundary Force and Displacemen

– Measurable!

Material: Stress and Strain

Stress - Strain Relations (Normal Stresses and Strains) ③

$$\varepsilon_{11} = \text{function}(\sigma_{11}) \quad ?$$



$$\varepsilon_{11} = \text{function}(\sigma_{11} \;\; \sigma_{22} \;\; \sigma_{33})$$
$$\varepsilon_{22} = \text{function}(\sigma_{11}, \; \sigma_{22} \;\; \sigma_{33})$$
$$\varepsilon_{ik} = \text{function}(\sigma_{je}) \qquad i, j, k, e \in \{1, 2, 3\}$$

Linear theory:

$$\varepsilon_{11} = S_{1111}\sigma_{11} + S_{1122}\sigma_{22} + S_{1133}\sigma_{33}$$
$$\varepsilon_{22} = S_{2211}\sigma_{11} + S_{2222}\sigma_{22} + S_{2233}\sigma_{33}$$
$$\varepsilon_{33} = S_{3311}\sigma_{11} + S_{3322}\sigma_{22} + S_{3333}\sigma_{33}$$

$$\varepsilon_i = \sum_j S_{ij}\sigma_j = S_{ij}\,\sigma_j$$
$$= S_i{}^j \sigma_j$$

In Matrix Form:

Compliance Matrix

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{pmatrix} = \begin{pmatrix} S_{1111} & S_{1122} & S_{1133} \\ S_{2211} & S_{2222} & S_{2233} \\ S_{3311} & S_{3322} & S_{3333} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{pmatrix}$$

strain vector components

stress vector components

$$\begin{pmatrix} \varepsilon \end{pmatrix}_i = \begin{pmatrix} S \end{pmatrix}_i{}^j \begin{pmatrix} \sigma \end{pmatrix}_j$$

$$\sigma_{11} = Q_{1111}\varepsilon_{11} + Q_{1122}\varepsilon_{22} + Q_{1133}\varepsilon_{33}$$
$$\sigma_{22} = Q_{21}\varepsilon_1 + \dots$$

$$\sigma_k = \sum_e Q_{ke}\,\varepsilon_e = Q_{ke}\,\varepsilon_e$$
$$= Q_k{}^e \varepsilon_e$$

Stiffness Matrix

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{pmatrix} = \begin{pmatrix} Q_{1111} & Q_{1122} & Q_{1133} \\ Q_{2211} & Q_{2222} & Q_{2233} \\ Q_{3311} & Q_{3322} & Q_{3333} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{pmatrix}$$

Volumetric Strain ④

$$\frac{V'}{V} = 1 + \frac{dV}{V} = \left(1 + \frac{u_1}{X_1}\right)\left(1 + \frac{u_2}{X_2}\right)\left(1 + \frac{u_3}{X_3}\right) \quad \Big/ \ln$$

$$\ln\left(1 + \frac{dV}{V}\right) = \ln\left(1 + \frac{u_1}{X_1}\right) + \ln\left(1 + \frac{u_2}{X_2}\right) + \ln\left(1 + \frac{u_3}{X_3}\right)$$
$$= \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \equiv \varepsilon_V$$

Uniaxial Stress in direction j:

$$\varepsilon_i = S_{ij}\,\sigma_j \quad (\text{no sum})$$

Engineering Constants:

Young's Modulus

$$\left.\begin{aligned} \varepsilon_j &= S_{jj}\,\sigma_j \quad (\text{no sum}) \\ \sigma_j &= E_j\,\varepsilon_j \quad (\text{no sum}) \end{aligned}\right\} \Rightarrow E_j = \frac{1}{S_{jj}} \quad (\text{no sum})$$

$$\varepsilon_i = -\nu_{ij}\,\varepsilon_j \quad (\text{ns})$$
$$S_{ij}\,\sigma_j = -\nu_{ij}\,\varepsilon_j \quad (\text{ns})$$

Poisson Ratio

$$S_{ij}\,E_j\,\varepsilon_j = -\nu_{ij}\,\varepsilon_j \quad (\text{ns}) \quad \Rightarrow \quad S_{ij} = -\frac{\nu_{ij}}{E_j}$$

Incompressible Material

$$\varepsilon_v = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 0$$

Isotropic Incompressible under Uniaxial stress in j

$$\varepsilon_j = S_{jj}\,\sigma_j$$

$$\varepsilon_k = -\nu_{kj}\,\varepsilon_j = -\nu_{kj}\,S_{jj}\,\sigma_j \qquad \text{(no sums)}$$

$$\varepsilon_\ell = -\nu_{\ell j}\,\varepsilon_j = -\nu_{kj}\,S_{jj}\,\sigma_j$$

$$\varepsilon_j + \varepsilon_k + \varepsilon_\ell = 0 \;\Rightarrow\; (1-\nu-\nu)\,S_{jj}\,\sigma_j = 0$$

$$\Rightarrow \nu = \tfrac{1}{2}$$

Isotropic Incompressible has Poisson Ratio of $\frac{1}{2}$

Total Differential of Displacement

$$du_i = \sum_j \frac{\partial u_i}{\partial x_j}\,dx_j = \frac{\partial u_i}{\partial x_j}\,dx_j = \varepsilon_{ij}\,dx_j = \varepsilon_i{}^j\,dx_j$$

$$du_1 = \varepsilon_{11}\,dx_1 + \varepsilon_{12}\,dx_2 + \varepsilon_{13}\,dx_3$$
$$du_2 = \varepsilon_{21}\,dx_1 + \varepsilon_{22}\,dx_2 + \varepsilon_{23}\,dx_3$$
$$du_3 = \varepsilon_{31}\,dx_1 + \varepsilon_{32}\,dx_2 + \varepsilon_{33}\,dx_3$$

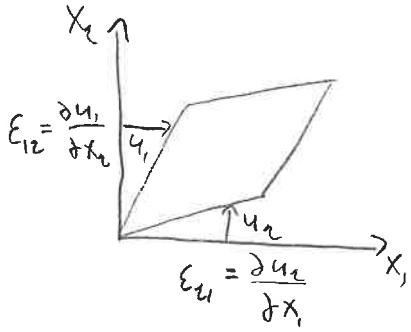
$$\begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$

Strain tensor – 9 components in 3-d space

Symmetric $\Rightarrow$ 6 Independent components

What are ε_{ij}, $i \neq j$?

Shear strains $\qquad \varepsilon_{ij} = \dfrac{\partial u_i}{\partial x_j}$



Why is the strain tensor symmetric?

Total Differential of force

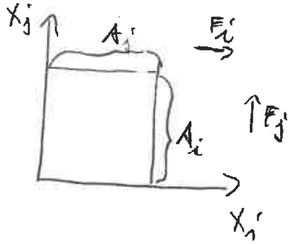
$$dF_i = \sum_j \frac{\partial F_i}{\partial A_j} dA_j = \sum \sigma_{ij}\, dA_j = \sigma_{ij}\, dA_j = \sigma_i{}^j\, dA_j$$

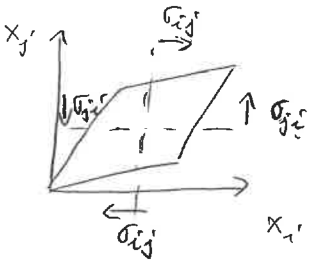
$$\begin{pmatrix} dF_1 \\ dF_2 \\ dF_3 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} dA_1 \\ dA_2 \\ dA_3 \end{pmatrix}$$

Stress tensor

symmetric – 6 independent components

Shear Stresses σ_{ij}, $i \neq j$





Why is the stress tensor symmetric?

Stress-Strain Relations

$$\varepsilon_{ij} = S_{ijkl}\, \sigma_{kl} \qquad \sigma_{ij} = Q_{ijkl}\, \varepsilon_{kl}$$

Compliance Matrix

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{32} \end{pmatrix} = \begin{pmatrix}
S_{1111} & S_{1122} & S_{1133} & S_{1112} & S_{1113} & S_{1121} & S_{1123} & S_{1131} & S_{1132} \\
S_{2211} & S_{2222} & S_{2233} & S_{2212} & S_{2213} & S_{2221} & S_{2223} & S_{2231} & S_{2232} \\
S_{3311} & S_{3322} & S_{3333} & S_{3312} & S_{3313} & S_{3321} & S_{3323} & S_{3331} & S_{3332} \\
S_{1211} & S_{1222} & S_{1233} & S_{1212} & S_{1213} & S_{1221} & S_{1223} & S_{1231} & S_{1232} \\
S_{1311} & S_{1322} & S_{1333} & S_{1312} & S_{1313} & S_{1321} & S_{1323} & S_{1331} & S_{1332} \\
S_{2111} & S_{2122} & S_{2133} & S_{2112} & S_{2113} & S_{2121} & S_{2123} & S_{2131} & S_{2132} \\
S_{2311} & S_{2322} & S_{2333} & S_{2312} & S_{2313} & S_{2321} & S_{2323} & S_{2331} & S_{2332} \\
S_{3111} & S_{3122} & S_{3133} & S_{3112} & S_{3113} & S_{3121} & S_{3123} & S_{3131} & S_{3132} \\
S_{3211} & S_{3222} & S_{3233} & S_{3212} & S_{3213} & S_{3221} & S_{3223} & S_{3231} & S_{3232}
\end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{21} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{32} \end{pmatrix}$$

81 components

Stiffness Matrix

$$\begin{matrix} \sigma_{11} \\ \sigma_{22} \\ \vdots \\ \vdots \end{matrix} \qquad \begin{matrix}
Q_{1111} & Q_{1122} & Q_{1133} & Q_{1112} & Q_{1113} & Q_{1121} & Q_{1123} & Q_{1131} & Q_{1132} \\
Q_{2211} & Q_{2222} & Q_{2233} & Q_{2212} & Q_{2213} & Q_{2221} & Q_{2223} & Q_{2231} & Q_{2232} \\
Q_{3311} & Q_{3322} & Q_{3333} & Q_{3312} & Q_{3313} & Q_{3321} & Q_{3323} & Q_{3331} & Q_{3332} \\
Q_{1211} & Q_{1222} & Q_{1233} & Q_{1212} & Q_{1213} & Q_{1221} & Q_{1223} & Q_{1231} & Q_{1232} \\
Q_{1311} & Q_{1322} & Q_{1333} & Q_{1312} & Q_{1313} & Q_{1321} & Q_{1323} & Q_{1331} & Q_{1332} \\
Q_{2111} & Q_{2122} & Q_{2133} & Q_{2112} & Q_{2113} & Q_{2121} & Q_{2123} & Q_{2131} & Q_{2132} \\
Q_{2311} & Q_{2322} & Q_{2333} & Q_{2312} & Q_{2313} & Q_{2321} & Q_{2323} & Q_{2331} & Q_{2332} \\
Q_{3111} & Q_{3122} & Q_{3133} & Q_{3112} & Q_{3113} & Q_{3121} & Q_{3123} & Q_{3131} & Q_{3132} \\
Q_{3211} & Q_{3222} & Q_{3233} & Q_{3212} & Q_{3213} & Q_{3221} & Q_{3223} & Q_{3231} & Q_{3232}
\end{matrix} \qquad \begin{matrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \vdots \end{matrix}$$

81 components

Symmetry of Stiffness and Compliance

Strain Energy Density $\hat{W} = \frac{1}{2} \varepsilon_{ij} \sigma_{ij}$

$$\frac{d\hat{W}}{d\varepsilon_{ij}} = \sigma_{ij} \qquad \frac{d\hat{W}}{d\sigma_{ij}} = \varepsilon_{ij}$$

on the other hand:

$$\hat{W} = \frac{1}{2} \left(S_{ijkl} \sigma_{kl} \right) \sigma_{ij}$$

$$\frac{d\hat{W}}{d\sigma_{mn}} = \frac{1}{2} S_{mnkl} \sigma_{kl} + \frac{1}{2} S_{ijmn} \sigma_{ij} = \frac{1}{2}\left(S_{mnij} + S_{ijmn}\right)\sigma_{ij}$$

But this equals $\varepsilon_{mn} = S_{mnij} \sigma_{ij}$

$$\Rightarrow S_{mnij} = S_{ijmn}$$

Similarly for Stiffness:

$$\hat{W} = \frac{1}{2} \varepsilon_{ij} \left(Q_{ijkl} \varepsilon_{kl} \right)$$

$$\frac{d\hat{W}}{d\varepsilon_{mn}} = \frac{1}{2} Q_{mnkl} \varepsilon_{kl} + \frac{1}{2} \varepsilon_{ij} Q_{ijmn} = \frac{1}{2}\left(Q_{mnij} + Q_{ijmn}\right)\varepsilon_{ij}$$

$$= \sigma_{mn} = Q_{mnij} \varepsilon_{ij}$$

$\Rightarrow \quad Q_{mnij} = Q_{ijmn} \quad \Rightarrow$ 45 Independent Components

Other symmetries of Compliance and Stiffness

Due to symmetry of strain tensor:

$$Q_{ijkl} = Q_{ijlk}$$

$$S_{ijkl} = S_{jikl}$$

Due to symmetry of stress tensor:

$$Q_{ijkl} = Q_{jikl}$$

$$S_{ijkl} = S_{ijlk}$$

$$Q_{ijkl} = Q_{jilk}$$

$$S_{ijkl} = S_{jilk}$$

$\Rightarrow$ Matrices can be written as 6x6 square matrices

$\sigma_{ij} = \sigma_{ji}$

Engineering Strain $e_{ij} = \varepsilon_{ij} + \varepsilon_{ji} = 2\varepsilon_{ij}$

Symmetry $\rightarrow$ 21 Independent Components

6x6 Compliance Matrix

(11)

ε_{11}	S_{1111}	S_{1122}	S_{1133}	$2\,S_{1112}$	$2\,S_{1113}$	$2\,S_{1123}$	σ_{11}
ε_{22}	S_{2211}	S_{2222}	S_{2233}	$2\,S_{2212}$	$2\,S_{2213}$	$2\,S_{2223}$	σ_{22}
ε_{33}	S_{3311}	S_{3322}	S_{3333}	$2\,S_{3312}$	$2\,S_{3313}$	$2\,S_{3323}$	σ_{33}
$2\,\varepsilon_{12}$	$2\,S_{1211}$	$2\,S_{1222}$	$2\,S_{1233}$	$4\,S_{1212}$	$4\,S_{1213}$	$4\,S_{1223}$	σ_{12}
$2\,\varepsilon_{13}$	$2\,S_{1311}$	$2\,S_{1322}$	$2\,S_{1333}$	$4\,S_{1312}$	$4\,S_{1313}$	$4\,S_{1323}$	σ_{13}
$2\,\varepsilon_{23}$	$2\,S_{2311}$	$2\,S_{2322}$	$2\,S_{2333}$	$4\,S_{2312}$	$4\,S_{2313}$	$4\,S_{2323}$	σ_{23}

6x6 Stiffness Matrix → 21 indep. compon

σ_{11}	Q_{1111}	Q_{1122}	Q_{1133}	Q_{1112}	Q_{1113}	Q_{1123}	ε_{11}
σ_{22}	Q_{2211}	Q_{2222}	Q_{2233}	Q_{2212}	Q_{2213}	Q_{2223}	ε_{22}
σ_{33}	Q_{3311}	Q_{3322}	Q_{3333}	Q_{3312}	Q_{3313}	Q_{3323}	ε_{33}
σ_{12}	Q_{1211}	Q_{1222}	Q_{1233}	Q_{1212}	Q_{1213}	Q_{1223}	$2\,\varepsilon_{12}$
σ_{13}	Q_{1311}	Q_{1322}	Q_{1333}	Q_{1312}	Q_{1313}	Q_{1323}	$2\,\varepsilon_{13}$
σ_{23}	Q_{2311}	Q_{2322}	Q_{2333}	Q_{2312}	Q_{2313}	Q_{2323}	$2\,\varepsilon_{23}$

Orthotropic Material, on-axis co-ordinates

(12)

- 2 perpendicular lines of reflection symmetry (2d)
- 3 perpendicular planes of reflection symmetry (3d)

- Shear stresses are not linked to normal strains
- Normal stresses are not linked to shear strains

- Shear strains are not linked to normal stresses
- Normal strains are not linked to shear stresses

Compliance Matrix – 9 independent components

ε_{11}	S_{1111}	S_{1122}	S_{1133}				σ_{11}
ε_{22}	S_{2211}	S_{2222}	S_{2233}				σ_{22}
ε_{33}	S_{3311}	S_{3322}	S_{3333}				σ_{33}
$2\,\varepsilon_{12}$				$4\,S_{1212}$			σ_{12}
$2\,\varepsilon_{13}$					$4\,S_{1313}$		σ_{13}
$2\,\varepsilon_{23}$						$4\,S_{2323}$	σ_{23}

Stiffness Matrix – 9 independent components

σ_{11}	Q_{1111}	Q_{1122}	Q_{1133}				ε_{11}
σ_{22}	Q_{2211}	Q_{2222}	Q_{2233}				ε_{22}
σ_{33}	Q_{3311}	Q_{3322}	Q_{3333}				ε_{33}
σ_{12}				Q_{1212}			$2\,\varepsilon_{12}$
σ_{13}					Q_{1313}		$2\,\varepsilon_{13}$
σ_{23}						Q_{2323}	$2\,\varepsilon_{23}$

Why are shear stresses not linked to normal strains, etc?

Any reflections wrt a symmetry plane changes the sign of one co-ordinate (or displacement)

=> Normal strains and stresses invariant in symmetry transformations $(\varepsilon_{11})' = \frac{\partial u_1'}{\partial x_1'} = \frac{-\partial u_1}{-\partial x_1} = \varepsilon_{11}$ (n

=> Shear strains and stresses change sign in symmetry transformations

$$(\varepsilon_{ij})' = \frac{\partial u_i'}{\partial x_j'} = -\frac{\partial u_1}{\partial x_j} = -\varepsilon_{ij}$$

=> Stiffnesses and compliances linking shear and normal strains/stresses should change sign in symmetry transformations. The only constant being equal to its negative is zero – this of course applies also to material constants!

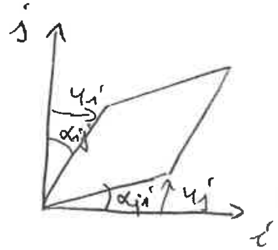
Orthotropic Material, continued

(13)

Why are shear stresses σ_{ij} not linked to shear strains σ_{jk}?

Reflection wrt symmetry plane (j,k) changes the sign of the co-ordinate (or displacement) i. Thus σ_{ij} changes sign, but σ_{jk} does not. The only material constant equal to its negative is again zero...

Measurement of Shear Strain



$$\varepsilon_{ij} = \varepsilon_{ji}$$

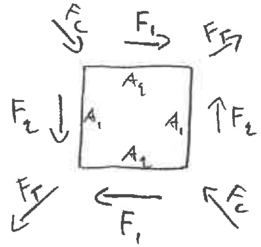
$$\varepsilon_{ij} = \frac{\partial u_i'}{\partial x_j'} = -\tan \alpha_{ij}'$$

$$\varepsilon_{ji} = \frac{\partial u_j'}{\partial x_i'} = \tan \alpha_{ji}'$$

$$\tan \alpha_{ij}' = -\tan \alpha_{ji}'$$

$$2\varepsilon_{ij} = \varepsilon_{ij} + \varepsilon_{ji} = -\tan\alpha_{ij}' + \tan\alpha_{ji}' = 2\tan|\alpha|$$

Measurement of Shear Stress



$$F_T = \cos 45^\circ F_1 + \cos(-45^\circ) F_2 = \frac{1}{\sqrt{2}} F_1 + \frac{1}{\sqrt{2}} F_2$$

$$A_T = \cos 45^\circ A_1 + \cos(-45^\circ) A_2 = \frac{1}{\sqrt{2}} A_1 + \frac{1}{\sqrt{2}} A_2$$

Set $A_1 = A_2$

$$\sigma_{12} = \sigma_{21} \Rightarrow \frac{\partial F_1}{\partial A_2} = \frac{\partial F_2}{\partial A_1} \Rightarrow F_1 = F_2$$

$$\sigma_T = \frac{F_T}{A_T} = \frac{F_1}{A_2} = \frac{F_2}{A_1} = \sigma_{12} = \sigma_{21} = \sigma_c$$

$$F_c = \cos(-45^\circ) F_1 + \cos 45^\circ F_2 = \sqrt{2} F_1 = \sqrt{2} F_2$$

$$A_c = \cos(-45^\circ) A_1 + \cos 45^\circ A_2 = \sqrt{2} A_1 = \sqrt{2} A_2$$

(14)

Shear Compliance in Orthotropy

$$S_{ijij} = \frac{\varepsilon_{ij}}{\sigma_{ij}} = \frac{\tan\alpha}{\sigma_T} = \frac{\tan\alpha}{\sigma_c} \qquad \text{(no sum)}$$

$$4 S_{ijij} = \frac{4\tan\alpha}{\sigma_T}$$

Shear Stiffness in Orthotropy

$$Q_{ijij} = \frac{\sigma_{ij}}{\varepsilon_{ij}} = \frac{\sigma_T}{\tan\alpha} = \frac{\sigma_c}{\tan\alpha}$$

Uniaxial Stress Experiment in j

Engineering Constants

$$\varepsilon_{ii} = S_{iijj}\,\sigma_{jj} \qquad \text{(no sum)}$$

1° $i \leq j$ $\qquad E_j = \frac{\sigma_{jj}}{\varepsilon_{jj}} \Rightarrow \boxed{E_j = \frac{1}{S_{jjjj}} \quad \text{(no sum)}}$

2° $i \neq j$

Poisson Ratio

$$\varepsilon_{ii} = -\nu_{ij}\,\varepsilon_{jj} = -\frac{\nu_{ij}}{E_j}\sigma_{jj}$$

$$\Rightarrow S_{iijj} = -\frac{\nu_{ij}}{E_j}$$

$$\Rightarrow \boxed{\nu_{ij} = \frac{-S_{iijj}}{S_{jjjj}} \quad \text{(no sum)}}$$

Symmetry of Compliance

$$S_{iijj} = S_{jjii}$$

$$\Rightarrow \boxed{\frac{\nu_{ij}}{E_j} = \frac{\nu_{ji}}{E_i}}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} ax+by \\ cx+dy \end{cases}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & z \\ y & k \end{pmatrix} = \begin{pmatrix} ax+by & az+bk \\ cx+dy & cz+dk \end{pmatrix}$$

Off-axis mechanical behavior of orthotropic materials

Off-axis stresses known – determine off-axis strains
1/ Stress transformation to on-axis
2/ Use on-axis compliance matrix to determine on-axis strains
3/ Negative transformation of strain to off-axis

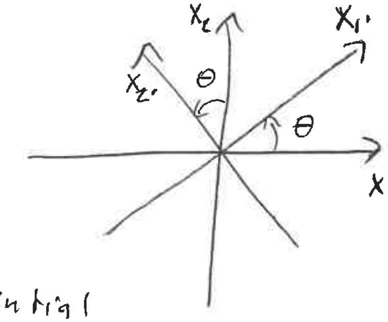
Off-axis strains known – determine off-axis stresses
1/ Strain transformation to on-axis
2/ Use on-axis stiffness matrix to determine on-axis stresses
3/ Negative transformation of stress to off-axis

This cannot be done using the Engineering constants!

Co-ordinate transformation of stress in two Dimensions

We are looking for a transformation matrix

$$\begin{pmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{1'2'} \end{pmatrix} = \begin{pmatrix} & M_\sigma & \end{pmatrix} \begin{matrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{matrix}$$

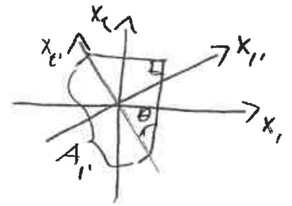


First, let us write the total differential of Force in X_1

$$dF_1 = \frac{\partial F_1}{\partial A_1} dA_1 + \frac{\partial F_1}{\partial A_2} dA_2$$

$$= \sigma_{11}\, d(\cos\theta\, A_{1'}) + \sigma_{12}\, d(\sin\theta\, A_{1'})$$

$$= \sigma_{11} \cos\theta\, dA_{1'} + \sigma_{12} \sin\theta\, dA_{1'}$$

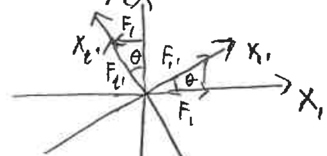


Let us then divide dF_1 into components

$$dF_1 = \cos\theta\, dF_{1'} - \sin\theta\, dF_{2'}$$

$$= \cos\theta\left[\sigma_{1'1'}\, dA_{1'} + \sigma_{1'2'}\, dA_{2'}\right]$$

$$- \sin\theta\left[\sigma_{2'2'}\, dA_{2'} + \sigma_{2'1'}\, dA_{1'}\right]$$



We find

$$\boxed{\begin{aligned} \frac{dF_1}{dA_{1'}} &= \sigma_{11}\cos\theta + \sigma_{12}\sin\theta \\ \frac{dF_1}{dA_{1'}} &= \sigma_{1'1'}\cos\theta - \sigma_{2'1'}\sin\theta \end{aligned}}$$

Let us then write the total differential of Force in x_2

$$dF_2 = \sigma_{21}\, dA_1 + \sigma_{22}\, dA_2 = \sigma_{21}\cos\theta\, dA_{1'} + \sigma_{22}\sin\theta\, dA_{1'}$$

And the components of dF_2

$$dF_2 = \sin\theta\, dF_{1'} + \cos\theta\, dF_{2'}$$

$$= \sin\theta\left[\sigma_{1'1'}\, dA_{1'} + \sigma_{1'2'}\, dA_{2'}\right] + \cos\theta\left[\sigma_{2'2'}\, dA_{2'} + \sigma_{2'1'}\, dA_{1'}\right]$$

We find

$$\boxed{\begin{aligned} \frac{dF_2}{dA_{1'}} &= \sigma_{21}\cos\theta + \sigma_{22}\sin\theta \\ \frac{dF_2}{dA_{1'}} &= \sigma_{1'1'}\sin\theta + \sigma_{2'1'}\cos\theta \end{aligned}}$$

Write

$\cos\theta \equiv m$

$\sin\theta \equiv n$

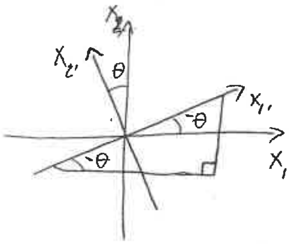
$$\boxed{\begin{aligned} m\sigma_{11} + n\sigma_{12} &= m\sigma_{1'1'} - n\sigma_{2'1'} \\ m\sigma_{21} + n\sigma_{22} &= n\sigma_{1'1'} + m\sigma_{2'1'} \end{aligned}}$$

Two Equations, two unknowns: solve $\sigma_{1'1'}$ and $\sigma_{2'1'}$

Result: $\sigma_{1'1'} = m^2\sigma_{11} + n^2\sigma_{22} + 2mn\sigma_{12}$

$$\sigma_{2'1'} = \sigma_{1'2'} = -mn\sigma_{11} + mn\sigma_{22} + (m^2-n^2)\sigma_{12}$$

Now, we are missing $\sigma_{2'2'}$



Total differential of F_1

$$dF_1 = \sigma_{11}\, dA_1 + \sigma_{12}\, dA_2$$

$$= \sigma_{11}\, d\left(\sin(-\theta) A_{2'}\right) + \sigma_{12}\, d\left(\cos(-\theta) A_{2'}\right)$$

$$= -\sigma_{11}\sin\theta\, dA_{2'} + \sigma_{12}\cos\theta\, dA_{2'}$$

Components

$$dF_1 = \cos\theta\, dF_{1'} - \sin\theta\, dF_{2'}$$

$$= \cos\theta\left[\sigma_{1'1'}\, dA_{1'} + \sigma_{1'2'}\, dA_{2'}\right] - \sin\theta\left[\sigma_{2'2'}\, dA_{2'} + \sigma_{2'1'}\, dA_{1'}\right]$$

$$\boxed{\begin{aligned} \frac{dF_1}{dA_{2'}} &= -\sigma_{11}\sin\theta + \sigma_{12}\cos\theta \\ \frac{dF_1}{dA_{2'}} &= \sigma_{1'2'}\cos\theta - \sigma_{2'2'}\sin\theta \end{aligned}}$$

$$-n\sigma_{11} + m\sigma_{12} = m\sigma_{1'2'} - n\sigma_{2'2'}$$

$$\Rightarrow \sigma_{2'2'} = \sigma_{11} - \frac{m}{n}\sigma_{12} + \frac{m}{n}\sigma_{1'2'}$$

$$= \sigma_{11} - \frac{m}{n}\sigma_{12} + \frac{m}{n}\left[-mn\sigma_{11} + mn\sigma_{22} + (m^2-n^2)\sigma_{12}\right]$$

$$= (1-m^2)\sigma_{11} + m^2\sigma_{22} + \left(-\frac{m}{n} + \frac{m^3}{n} - mn\right)\sigma_{12}$$

$$= n^2\sigma_{11} + m^2\sigma_{22} + \frac{m}{n}(-1-m^2-n^2)\sigma_{12}$$

$$= \frac{m}{n}(-2n^2)\sigma_{12}$$

$$= n^2\sigma_{11} + m^2\sigma_{22} - 2mn\sigma_{12}$$

All components known:

$$\begin{pmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \end{pmatrix} = \begin{pmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}$$

So, we have made a crd-transformation of stress. Was that right?

(18)

Inverse transformation should recover the initial state:

$$M^{-1} M = \mathbb{1}$$

M^{-1}: $(m)^{-1} = m$

$(n)^{-1} = -n$

$$M_\sigma^{-1} = \begin{pmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2-n^2 \end{pmatrix}$$

$$M^{-1} M = \begin{pmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2-n^2 \end{pmatrix} \begin{pmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{pmatrix} =$$

$$\begin{pmatrix} (m^2+n^2)^2 & 0 & 0 \\ 0 & (m^2+n^2)^2 & 0 \\ 0 & 0 & (m^2+n^2)^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \%$$

(19)

$$\varepsilon_{ij} = \frac{\partial u_i}{\partial x_j} \qquad \varepsilon_{i'j'} = \frac{\partial u_{i'}}{\partial x_{j'}}$$

Chain rule of partial derivatives:

$$d\varepsilon_{ij} = \frac{\partial \varepsilon_{ij}}{\partial x_k} dx_k$$

Small-Strain Theory; $\varepsilon_{i'j'} = \frac{\partial u_{i'}}{\partial x_{j'}} = \frac{\partial u_{i'}}{\partial x_k} \frac{\partial x_k}{\partial x_{j'}}$

$$\varepsilon_{1'1'} = \frac{\partial u_{1'}}{\partial x_1} \frac{\partial x_1}{\partial x_{1'}} + \frac{\partial u_{1'}}{\partial x_2} \frac{\partial x_2}{\partial x_{1'}}$$

$$\varepsilon_{1'2'} = \frac{\partial u_{1'}}{\partial x_1} \frac{\partial x_1}{\partial x_{2'}} + \frac{\partial u_{1'}}{\partial x_2} \frac{\partial x_2}{\partial x_{2'}}$$

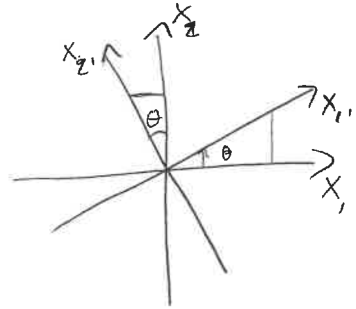
...

$$\varepsilon_{1'1'} = \frac{\partial u_{1'}}{\partial x_1} m + \frac{\partial u_{1'}}{\partial x_2} n$$

$$= \left[\frac{\partial u_{1'}}{\partial u_1} \frac{\partial u_1}{\partial x_1} + \frac{\partial u_{1'}}{\partial u_2} \frac{\partial u_2}{\partial x_1}\right] m + \left[\frac{\partial u_{1'}}{\partial u_1} \frac{\partial u_1}{\partial x_2} + \frac{\partial u_{1'}}{\partial u_2} \frac{\partial u_2}{\partial x_2}\right] n$$

$$= \left[m\varepsilon_{11} + n\varepsilon_{21}\right] m + \left[m\varepsilon_{12} + n\varepsilon_{22}\right] n$$

$$= m^2 \varepsilon_{11} + n^2 \varepsilon_{22} + 2mn\, \varepsilon_{12}$$



$x_{1'} = \cos\theta\, x_1 + \sin\theta\, x_2$

$x_{2'} = -\sin\theta\, x_1 + \cos\theta\, x_2$

$x_1 = \cos\theta\, x_{1'} - \sin\theta\, x_{2'}$

$x_2 = \sin\theta\, x_{1'} + \cos\theta\, x_{2'}$

Co-ordinate transformation of strain in two dimensions Continued

$$\varepsilon_{2'2'} = \frac{\partial u_{2'}}{\partial x_1}\frac{\partial x_1}{\partial x_{2'}} + \frac{\partial u_{2'}}{\partial x_2}\frac{\partial x_2}{\partial x_{2'}}$$

$$= \frac{\partial u_{2'}}{\partial x_1}(-n) + \frac{\partial u_{2'}}{\partial x_2} m$$

$$= \left[\frac{\partial u_{2'}}{\partial u_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial u_{2'}}{\partial u_2}\frac{\partial u_2}{\partial x_1}\right](-n) + \left[\frac{\partial u_{2'}}{\partial u_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_{2'}}{\partial u_2}\frac{\partial u_2}{\partial x_2}\right] m$$

$$= \left[(-n)\varepsilon_{11} + m\,\varepsilon_{21}\right](-n) + \left[-n\,\varepsilon_{12} + m\,\varepsilon_{22}\right] m$$

$$= n^2 \varepsilon_{11} + m^2 \varepsilon_{22} - 2mn\,\varepsilon_{12}$$

$$\varepsilon_{1'2'} = \frac{\partial u_{1'}}{\partial x_1}\frac{\partial x_1}{\partial x_{2'}} + \frac{\partial u_{1'}}{\partial x_2}\frac{\partial x_2}{\partial x_{2'}} = \left[\frac{\partial u_{1'}}{\partial u_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial u_{1'}}{\partial u_2}\frac{\partial u_2}{\partial x_1}\right](-n) +$$

$$\left[\frac{\partial u_{1'}}{\partial u_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_{1'}}{\partial u_2}\frac{\partial u_2}{\partial x_2}\right] m = \left[m\,\varepsilon_{11} + n\,\varepsilon_{21}\right](-n) + \left[m\,\varepsilon_{12} + n\,\varepsilon_{22}\right] m$$

$$= -mn\,\varepsilon_{11} + mn\,\varepsilon_{22} + (m^2 - n^2)\,\varepsilon_{12}$$

$$-nm\varepsilon_{11} + mn\varepsilon_{22} + m^2\varepsilon_{12} - n^2\varepsilon_{21}$$

$$\begin{pmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{1'2'} \end{pmatrix} = \begin{pmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{pmatrix}$$

Co-ordinate transformation of Compliance

1/ Transform stress to on-axis
2/ Use on-axis compliance to compute strain
3/ Transform strain to off-axis
4/ Identify off-axis Compliance components – they are given as polynomials of on-axis compliance components!

Co-ordinate transformation of Stiffness

1/ Transform strain to on-axis
2/ Use on-axis stiffness to compute stress
3/ Transform stress to off-axis
4/ Identify off-axis Stiffness components – they are given as polynomials of on-axis stiffness components!

How do we determine Compliance Matrix experimentally?

Once we know Compliance Matrix, how do we get to know Stiffness Matrix?

If we know Stiffness Matrix, how do we get to know Compliance Matrix?

$$\left.\begin{array}{l}\sigma = Q\,\varepsilon \\ \varepsilon = S\,\sigma\end{array}\right\} \quad \begin{array}{lll}\varepsilon = S Q \varepsilon & \Rightarrow \quad SQ = \mathbb{1} & \Rightarrow Q = S^{-1} \\ \sigma = Q S \sigma & \Rightarrow \quad QS = \mathbb{1} & \Rightarrow S = Q^{-1}\end{array}$$

How do we invert a matrix?

Gaussian Elimination:

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \quad \Big| \quad A^{-1}(\;)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$ax + by = x'$$
$$cx + dy = y' \quad \Big| \cdot \left(-\frac{a}{c}\right)$$
$$\left(b - \frac{ad}{c}\right) y = x' - \frac{a}{c} y'$$

$$\begin{pmatrix} a & b \\ 0 & b - \frac{ad}{c} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -\frac{a}{c} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$ax + by = x' \quad \Big| \cdot \left(-\frac{b - \frac{ad}{c}}{b}\right)$$
$$\left(b - \frac{ad}{c}\right) y = x' - \frac{a}{c} y'$$
$$-\frac{a}{b}\left(b - \frac{ad}{c}\right) x = \left(1 - 1 + \frac{ad}{bc}\right) x' - \frac{a}{c} y'$$

$$\begin{pmatrix} -\frac{a}{b}\left(b - \frac{ad}{c}\right) & 0 \\ 0 & b - \frac{ad}{c} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{ad}{bc} & -\frac{a}{c} \\ 1 & -\frac{a}{c} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{ad}{bc}\left(-\frac{b}{a}\frac{1}{b - \frac{ad}{c}}\right) & \frac{a}{c}\,\frac{b}{a}\,\frac{1}{b - \frac{ad}{c}} \\ \frac{1}{b - \frac{ad}{c}} & -\frac{a}{c}\,\frac{1}{b - \frac{ad}{c}} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{d}{c}\,\frac{1}{b - \frac{ad}{c}} & \frac{b}{c}\,\frac{1}{b - \frac{ad}{c}} \\ \frac{c}{bc - ad} & -\frac{a}{bc - ad} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-d}{bc - ad} & \frac{b}{bc - ad} \\ \frac{c}{bc - ad} & \frac{-a}{bc - ad} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

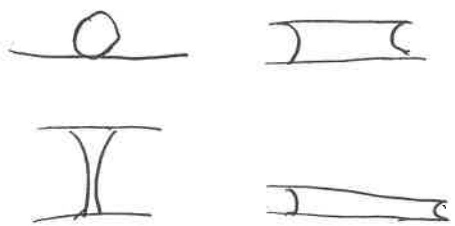
Verify if:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \%$$

What if $(ad-bc)=0$?

What is the Young's Modulus of Water?

Negative in compression?

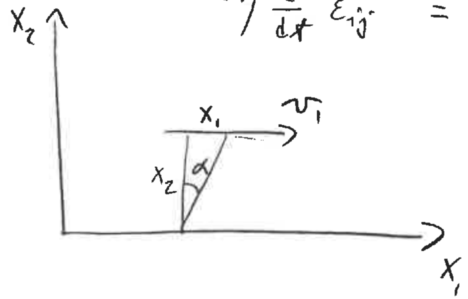


$$\sigma_{ij} = \text{function}\left(\dot{\varepsilon}_{kl}\right) = \text{function}\left(\frac{d\varepsilon_{kl}}{dt}\right)$$

Newtonian Viscosity for Isotropic Fluid:

$$\sigma_{ij} = \eta \dot{\varepsilon}_{ij} = \eta \frac{dv_i}{dx_j} = \eta \frac{d^2 u_i}{dt\, dx_j} = \eta \frac{d}{dt} \tan\alpha$$

$$= \eta \frac{d}{dt} \varepsilon_{ij} = \eta \dot{\varepsilon}_{ij}$$



Time-dependent Mechanical Behavior – Linear Viscoelasticity

$$\varepsilon_{ij} = S_{ijkl}\, \sigma_{kl}$$

$$\Rightarrow \quad d\varepsilon_{ij}(t) = C_{ijkl}(t-\tau)\, d\sigma_{kl}(\tau)$$

$$\varepsilon_{ij}(t) = \int_{\sigma(\tau=-\infty)}^{\sigma(\tau=t)} C_{ijkl}(t-\tau)\, d\sigma_{kl}(\tau)$$

$$= \int_{-\infty}^{t} C_{ijkl}(t-\tau) \frac{d\sigma_{kl}}{d\tau}\, d\tau$$

$C_{ijkl}(t) \equiv$ Creep Compliance

$$\sigma_{ij} = Q_{ijkl}\, \varepsilon_{kl}$$

$$\Rightarrow \quad d\sigma_{ij}(t) = R_{ijkl}(t-\tau)\, d\varepsilon_{kl}(\tau)$$

$$\sigma_{ij}(t) = \int_{-\infty}^{t} R_{ijkl}(t-\tau) \frac{d\varepsilon_{kl}(\tau)}{d\tau}\, d\tau$$

$R_{ijkl}(t) \equiv$ Relaxation Modulus

What kind of a function is the Creep Compliance?

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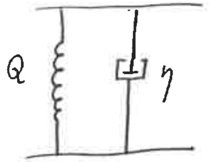
Voigt Element



$$\sigma = \varepsilon Q + \dot{\varepsilon}\eta$$

$$\frac{d\sigma}{dt} = 0 \Rightarrow \dot{\varepsilon} Q + \ddot{\varepsilon}\eta = 0$$

write $\dot{\varepsilon} \equiv a \Rightarrow aQ + \dot{a}\eta = 0$

$$aQ = -\frac{da}{dt}\eta$$

$$-\frac{Q}{\eta}dt = \frac{da}{a} \quad \Big| \int$$

$$-\frac{Q}{\eta}t = \ln a + C \quad \Big| \exp()$$

$$a = \dot{\varepsilon} = C_2 e^{-\frac{Q}{\eta}t} \approx \frac{\sigma}{\eta}e^{-\frac{t}{\tau}}$$

$\tau \equiv$ Retardation Time

$$\frac{d\varepsilon}{dt} = \frac{\sigma}{\eta}e^{-\frac{t}{\tau}}$$

$$d\varepsilon = \frac{\sigma}{\eta}e^{-\frac{Q}{\eta}t}\,dt \quad \Big| \int$$

$$\varepsilon = -\frac{\sigma}{Q}e^{-\frac{t}{\tau}} + C_3$$

$$\varepsilon(t=0) = -\frac{\sigma}{Q} + C_3 = 0 \Rightarrow C_3 = \frac{\sigma}{Q}$$

$$\varepsilon = \frac{\sigma}{Q}\left(1 - e^{-t/\tau}\right) \Rightarrow \boxed{C(t) = C_\infty\left(1 - e^{-t/\tau}\right)}$$

$$\varepsilon = C\sigma = \sigma\left[\frac{1}{Q}\left(1 - e^{-t/\tau}\right)\right]$$

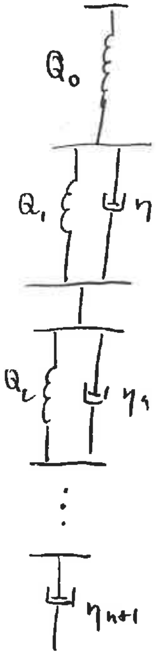
Voigt Elements in Series:

$$\varepsilon = \sum_i (C_\infty)_i \left(1 - e^{-t/\tau_i}\right)\sigma$$

$$C(t) = \int_0^\infty (C_\infty)_i \left(1 - e^{-t/\tau_i}\right) di = \int_0^\infty C_\infty(\tau)\left(1 - e^{-t/\tau}\right)\frac{di}{d\tau}\,d\tau$$

What if two of the Voigt elements are degenerate?

28



$$C(t) = C_0 + \int_0^\infty (C_\infty)_i \left(1 - e^{-t/\tau}\right)\frac{di}{d\tau}\,d\tau + \frac{t}{\eta}$$

$$= \boxed{C_0} + \int_{-\infty}^{\infty} \boxed{L(\tau)} \left(1 - e^{-t/\tau}\right) d(\ln\tau) + \frac{t}{\boxed{\eta}}$$

Glassy Compliance

Retardation Spectrum

Steady-state Viscosity

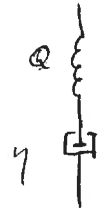
Steady-state Compliance?
Equilibrium
Rubbery

only if $\eta \to \infty$

$$C(\infty) = C_0 + \int_{-\infty}^{\infty} L(\tau)\, d(\ln\tau)$$

What kind of a function is the Relaxation Modulus?

Maxwell Element



$$\sigma = Q\varepsilon_1 = \eta\dot{\varepsilon}_2$$

$$\varepsilon = \varepsilon_1 + \varepsilon_2$$

$$\frac{d\varepsilon}{dt} = \frac{d\varepsilon_1}{dt} + \frac{d\varepsilon_2}{dt} = 0$$

$$\frac{1}{Q}\frac{d\sigma}{dt} + \frac{\sigma}{\eta} = 0$$

$$\frac{d\sigma}{\sigma} = -\frac{Q}{\eta}dt$$

$$\ln\sigma = -\frac{Q}{\eta}t + C$$

$$\sigma = C_2\, e^{-\frac{Q}{\eta}t} = R_0\,\varepsilon\, e^{-t/\tau}$$

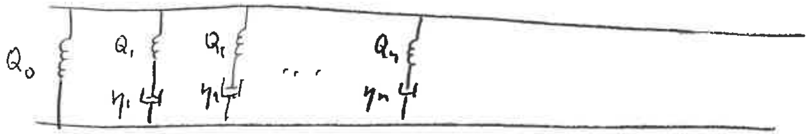
$$R(t) = R_0\, e^{-t/\tau}$$

Maxwell elements in parallell:

$$\sigma = \sum_i \sigma_i = \sum_i R_i(t)\,\varepsilon = \varepsilon \sum_i (R_0)_i\, e^{-t/\tau_i}$$

$$R(t) = \int_i (R_0)_i\, e^{-t/\tau_i}\, di = \int_0^\infty (R_0)_i\, e^{-t/\tau_i}\frac{di}{d\tau}\, d\tau$$

What if some of the elements are degenerate?



$$R(t) = Q_0 + \sum_i (R_0)_i\, e^{-t/\tau_i}$$

$$= Q_0 + \int_0^\infty (R_0)_i\, e^{-t/\tau_i}\frac{di}{d\tau}\, d\tau$$

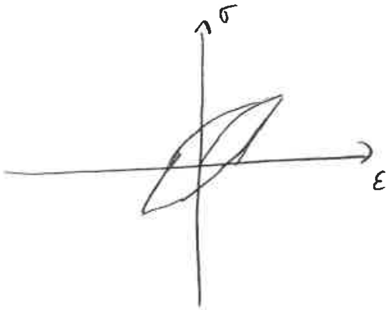
$$= \boxed{R_\infty} + \int_{-\infty}^{\infty} \boxed{H(\tau)}\, e^{-t/\tau}\, d(\ln\tau)$$

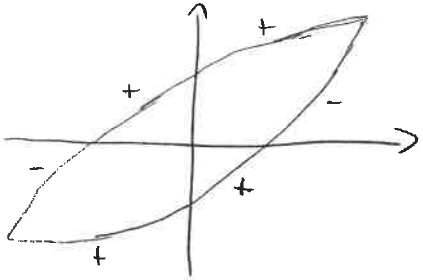
Equilibrium Modulus   Relaxation Spectrum

Glassy Modulus:

$$R(t=0) = R_\infty + \int_{-\infty}^{\infty} H(\tau)\, d(\ln\tau) \equiv R_0$$

Cyclical Experiment





Strain Energy Density

$$\dot{W} = \frac{1}{2}\sigma\varepsilon = \frac{1}{2} Q \varepsilon^2$$

$$\frac{d\dot{W}}{d\varepsilon} = \sigma$$

Where does the work (energy) go?

Stress – Strain – Time – Temperature – Moisture - relations

Crosslinked polymers:

Equilibrium Elasticity exist

$$\sigma = \varepsilon\left[R_\infty + \sum_{i} (R_0)_i \, e^{-t/\tau_i}\right]$$

Noncrosslinked: Liquid-like flow $(R_\infty = 0)$

Liquid-like flow

$$\varepsilon = \sigma\left[\quad \dots + t/\eta\right] \qquad \eta < \infty$$

Amorphous Polymers:

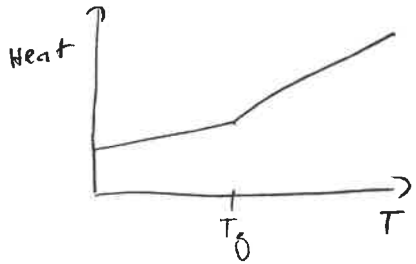
1/ Large-deformation Equilibrium properties

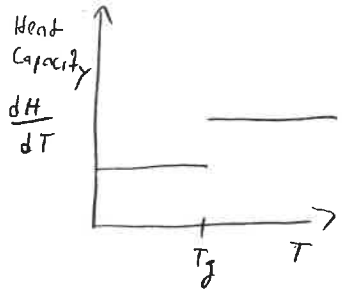
2/ Small-deformation nonequilibrium properties (viscoelastic)

3/ Large-deformation time-dependent properties

Amorphous Polymers:

Glass Transition - second-order transition

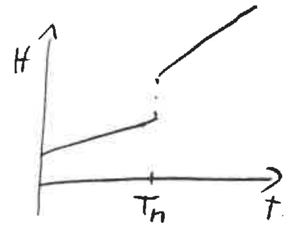




Crystalline polymers:

Melting - first-order transition

First-order transition



1/ Large-deformation Equilibrium properties

From kinetic theory:

$$\sigma = \rho k T \left(\varepsilon - \varepsilon^{-2}\right)$$

Eng. tensile stress
— " — strain

Shear Modulus $G = \rho k T$

How to approach this behavior?

- increase time
  = reduce straining rate
- speed up relaxation (temperature, moisture, ...)

Time-Temperature Equivalency

- All characteristic times similarly affected by temperature change

Thermorheologically simple materials:

$$C(T,t) = C\left(T_0, \frac{t}{a(T)}\right)$$

$$T > T_0 \Rightarrow t < \frac{t}{a(T)}$$

$$\Rightarrow a(T) < 1$$

$$T < T_0 \Rightarrow t > \frac{t}{a(T)}$$

$$\Rightarrow a(T) > 1$$

$$\boxed{\text{Reduced time} \equiv \frac{t}{a(T)}}$$

$$R(T,t) = R\left(T_0, \frac{t}{b(T)}\right)$$

$$b(T) \approx a(T) \quad (?)$$

WLF:

$$\log a(T) = -\frac{C_1 (T-T_0)}{C_2 + T - T_0} \approx \frac{-8{,}86\,(T-T_0)}{101{,}6 + T - T_0}$$

$$\log a = \frac{\ln a}{\ln 10}$$

$$10^{\log a} = a = 10^{-\frac{C_1(T-T_0)}{C_2 + T - T_0}}$$

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Principal Stresses – stress Eigenvalues

$$A x = \lambda x \Rightarrow (A - \lambda \mathbb{1}) x = 0$$

nontrivial solution $\Rightarrow$ $|A - \lambda \mathbb{1}| = 0$

{Characteristic / Secular} determinant of the stress tensor = 0

$$(\sigma_{ij} - \sigma \delta_{ij}) n_j = 0$$

$$\begin{vmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma \end{vmatrix} = 0$$

$$\begin{pmatrix} F \end{pmatrix} = \begin{pmatrix} \Sigma \end{pmatrix} \begin{pmatrix} A \end{pmatrix}$$

$$F = \Sigma A$$

$$\boxed{S^{-1} F} = S^{-1} \Sigma A$$

$$= \boxed{S^{-1} \Sigma S} \boxed{S^{-1} A}$$

3rd degree polynomial → 3 solutions

Hermitian matrix → 3 real roots

3 stress Eigenvalues : $\sigma_1 \quad \sigma_2 \quad \sigma_3$

Stress tensor can be diagonalized by Unitary Transformation → Stresses in Normal Co-ordinates

$$S^{-1} \Sigma S = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

$$B \bar{x} = 0 \quad | \; B^{-1}(\;)$$

$$B^{-1} B \bar{x} = 0$$

$$\mathbb{1} \bar{x} = \bar{x}$$

$$B^{-1} = \frac{1}{\det(B)} B^{T} \qquad \det B = 0$$

$$\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \sigma_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \sigma_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda \mathbb{1} =$$

$$\lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \sigma_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} S_{11} \\ S_{21} \\ S_{31} \end{pmatrix}$$

$$S^{-1} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} S_{21} \\ S_{22} \\ S_{32} \end{pmatrix}$$

$$S^{-1} \bar{Y} = \bar{X}$$

$$\bar{Y} = S\bar{X}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} S_{13} \\ S_{23} \\ S_{33} \end{pmatrix}$$

The third-degree polynomial can be given in terms of stress Invariants

$$\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0$$

$$I_1 = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$I_2 = \frac{1}{2} (\sigma_{ii} \sigma_{jj} - \sigma_{ij} \sigma_{ij})$$

$$= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2$$

$$I_3 = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{33} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix}$$

The stress Invariants can be given in a co-ordinate system where the stress tensor is diagonal
- in terms of principal stresses

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

$$I_3 = \begin{vmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{vmatrix} = \sigma_1\sigma_2\sigma_3$$

How do we determine the direction of a principal stress?

Once solving the stress Eigenvalues, solve the Eigenvalue Equation for Eigenvectors:

$$\Sigma x = \sigma_i x$$

or

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \sigma_i \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Where the values of x_1/x_2 and $\frac{x_1}{x_3}$ can be found.

Normalized Eigenvector: $x_1^2 + x_2^2 + x_3^2 = 1$

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$$[A]\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} = \begin{matrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 & a_{11}y_1 + a_{12}y_2 + a_{13}y_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 & \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 & \end{matrix}$$

$$= \begin{pmatrix} \lambda_1 x_1 & \lambda_2 y_1 & \lambda_3 z_1 \\ \lambda_1 x_2 & \lambda_2 y_2 & \lambda_3 z_2 \\ \lambda_1 x_3 & \lambda_2 y_3 & \lambda_3 z_3 \end{pmatrix} =$$

$$= \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$[A][U] = [U]\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$$

$$[U^{-1}][A][U] = [\lambda]$$

Elastoplasticity

$$\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{p}$$

$$\sigma_{ij} = Q_{ijkl}\,\varepsilon_{kl}^{e} = Q_{ijkl}\,(\varepsilon_{kl} - \varepsilon_{kl}^{p})$$

Yield Function $f = f(\sigma_{ij})$

Yield condition $f=0 \Rightarrow$ yield surface

Isotropic Hardening - yield surface expands

Kinematic Hardening - yield surface translates

Ideal Plasticity - yield surface stationary in stress space

No stress state can exist outside of the yield surface:

$f<0$: Elastic range

$f=0$: Plastic deformation may occur

Yield surface gradient:

- in co-ordinate space (3-d) $\nabla f = \bar{e}_r \frac{\partial}{\partial x_r} f$

- in stress space (9-d) $\nabla f = \bar{e}_{ij} \frac{\partial}{\partial \sigma_{ij}} f$

Gradient component: $\frac{\partial f}{\partial \sigma_{ij}}$

Plastic loading: $f = 0, \quad \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} > 0$

$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} < 0 \Rightarrow$ Elastic Unloading

$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = 0 \Rightarrow$ Neutral Loading

Example of yield criterion,

formulated in 3-d principal stress space:

Von Mises $f = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] - k^2 = 0.$

$k \equiv$ shear yield stress $\quad \sigma_Y = \sqrt{3}\,k$

The von Mises criterion can be given in terms of Deviatoric Stresses

$$S_{ij} \equiv \sigma_{ij} - \frac{1}{3}\sigma_{kk}\,\delta_{ij}$$

or

$$\begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} = \begin{pmatrix} \sigma_{11} - \frac{\sigma_{kk}}{3} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \frac{\sigma_{kk}}{3} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \frac{\sigma_{kk}}{3} \end{pmatrix}$$

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The principal deviatoric stresses can be determined through the Eigenvalue Equation

$$S x = s x \quad \Rightarrow \quad (S - s\mathbb{1}) x = 0$$

$$\Rightarrow \det(S - s\mathbb{1}) = 0$$

$$\begin{vmatrix} S_{11}-s & S_{12} & S_{13} \\ S_{21} & S_{22}-s & S_{23} \\ S_{31} & S_{32} & S_{33}-s \end{vmatrix} = 0$$

$$s^3 - J_1 s^2 - J_2 s - J_3 = 0$$

$$J_1 = S_{11} + S_{22} + S_{33} = S_{ii} = 0$$

$$J_2 = -\left(S_{11}S_{22} + S_{22}S_{33} + S_{33}S_{11} - S_{12}^2 - S_{13}^2 - S_{23}^2\right) = -\frac{1}{2}\left(S_{ii}S_{jj} - S_{ij}S_{ij}\right)$$

$$= \frac{1}{2} S_{ij} S_{ij}$$

$$J_3 = \det S = \frac{1}{6} \varepsilon_{ijk} \varepsilon_{pqr} S_{ip} S_{jq} S_{kr}$$

$$= \begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{vmatrix}$$

Levi-Civita symbol ε_{ijk}

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The deviatoric stress Invariant

$$J_2 =$$

$$= -\left(S_{11}S_{22} + S_{22}S_{33} + S_{33}S_{11} - S_{12}^2 - S_{13}^2 - S_{23}^2\right)$$

$$= -\Big[(\sigma_{11}-m)(\sigma_{22}-m) + (\sigma_{22}-m)(\sigma_{33}-m) + (\sigma_{33}-m)(\sigma_{11}-m) - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2\Big] \qquad m \equiv \frac{\sigma_{ii}}{3}$$

$$= -\Big[\sigma_{11}\sigma_{22} - \sigma_{11} m - \sigma_{22} m + m^2 + \dots\Big]$$

$$= -\Big[\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - 2m(\sigma_{11}+\sigma_{22}+\sigma_{33}) + 3\left(\frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3}\right)^2 + \dots\Big]$$

$$= -\Big[\qquad \dots - \frac{2}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33})^2 + \frac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33})^2 + \dots\Big]$$

$$= \frac{1}{6}\Big[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2\Big] + \sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2$$

Change to a basis where Σ is diagonal

$$J_2 = \frac{1}{6}\Big[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\Big]$$

$\Rightarrow$ Von Mises Criterion becomes

$$f = J_2 - k^2 = 0 \qquad !$$

Incremental Plasticity

Drucker postulate:

$$(\sigma_{ij} - \sigma_{ij}^{0})\, d\varepsilon_{ij}^{p} + d\sigma_{ij}\, d\varepsilon_{ij}^{p} \geq 0$$

where σ_{ij} on the yield surface

σ_{ij}^{0} on or inside the yield surface

σ_{ij}^{0} may equal σ_{ij} $\Rightarrow$ $d\sigma_{ij}\, d\varepsilon_{ij}^{p} \geq 0$

σ_{ij} may be $\gg \sigma_{ij}^{0}$ $\Rightarrow$ $(\sigma_{ij} - \sigma_{ij}^{0})\, d\varepsilon_{ij}^{p} \geq 0$

$\Rightarrow$ yied surface is convex

$d\sigma_{ij}\, d\varepsilon_{ij}^{p} \geq 0$

$d\sigma_{ij}$ limited by any tangent of the yield surface

$\Rightarrow$ $d\varepsilon_{ij}^{p}$ is normal to the yield surface

$$d\varepsilon_{ij}^{p} = d\lambda \frac{\partial f}{\partial \sigma_{ij}'} \qquad df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = \frac{\partial f}{\partial \sigma_{ij}'} \frac{d\sigma_{ij}'}{d\lambda} d\lambda$$

$$= d\varepsilon_{ij}^{p} \frac{d\sigma_{ij}}{d\lambda}$$

Insert Von Mises

$$d\varepsilon_{ij}^{p} = d\lambda \frac{\partial f}{\partial \sigma_{ij}'} = d\lambda \frac{\partial (J_2 - k^2)}{\partial \sigma_{ij}}$$

$$= d\lambda \frac{\partial \left(\frac{1}{2} S_{ij} S_{ij} - k^2\right)}{\partial S_{ij}} \frac{\partial S_{ij}}{\partial \sigma_{ij}}$$

$$\frac{\partial S_{ij}}{\partial \sigma_{ij}} = \frac{\partial(\sigma_{ij} - m\delta_{ij})}{\partial \sigma_{ij}} = 1$$

$$\frac{d S_{ij}}{d \sigma_{ij}} = \frac{d\left[\sigma_{ij}\left(1 - \frac{\delta_{ij}}{3}\right)\right]}{d\sigma_{ij}} = \frac{2}{3}$$

$$= d\lambda\, S_{ij}$$

Suppose a universal stress-strain-curve

$$\bar{\sigma} = h\left(\int d\bar{\varepsilon}^{p}\right) = h(\bar{\varepsilon}^{p})$$

Define Effective stress $\bar{\sigma} \equiv (3 J_2)^{\frac{1}{2}} = \left(\frac{3}{2} S_{ij} S_{ij}\right)^{\frac{1}{2}}$

$\Rightarrow$ for uniaxial stress σ_{11}

$$\bar{\sigma} = \left(\frac{3}{2}\right)^{\frac{1}{2}} \left[\left(\frac{2}{3}\right)^2 \sigma_{11}^2 + \left(-\frac{1}{3}\right)^2 \sigma_{11}^2 + \left(-\frac{1}{3}\right)^2 \sigma_{11}^2\right]^{\frac{1}{2}} = \left(\frac{3}{2} \frac{6}{9} \sigma_{11}^2\right)^{\frac{1}{2}} = |\sigma_{11}|$$

for pure shear σ_{12} $\quad \bar{\sigma} = \left(\frac{3}{2} 2\sigma_{12}^2\right)^{\frac{1}{2}} = \sqrt{3}\, |\sigma_{12}|$

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$$S_{ij}' = S_{ij}'(\sigma_{kl}, m) = \sigma_{ij} - m\delta_{ij}$$

$$dS_{ij}' = \frac{\partial S_{ij}'}{\partial \sigma_{kl}} d\sigma_{kl} + \frac{\partial S_{ij}'}{\partial m} dm$$

$$\frac{\partial S_{ij}'}{\partial \sigma_{kl}} = \delta_{ik}\delta_{jl}$$

$$\frac{\partial S_{ij}'}{\partial m} = -\delta_{ij}$$

$$m = m(\sigma_{kl})$$

$$dm = \frac{\partial m}{\partial \sigma_{kl}} d\sigma_{kl}$$

$$\frac{\partial m}{\partial \sigma_{kl}} = \frac{1}{3}\delta_{kl}$$

$$dS_{ij}' = \delta_{ik}\delta_{jl}\, d\sigma_{kl} - \frac{1}{3}\delta_{kl}\, d\sigma_{kl}$$

$$\frac{dS_{ij}'}{d\sigma_{kl}} = \delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}$$

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Define effective plastic strain Increment

$$d\bar{\varepsilon}^p \equiv \frac{1}{2}\left(d\varepsilon_{ij}^{p\prime}\, d\varepsilon_{ij}^{p\prime}\right)^{\frac{1}{2}} = \left(\frac{1}{4}\, d\varepsilon_{ij}^{p\prime}\, d\varepsilon_{ij}'\right)^{\frac{1}{2}}$$

For uniaxial stress σ_{11}

$$d\varepsilon_{11}^p = d\lambda \frac{\partial f}{\partial \sigma_{11}} = d\lambda \frac{1}{6}\frac{\partial\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{33}-\sigma_{11})^2\right]}{\partial \sigma_{11}}$$

$$= d\lambda\, \frac{1}{6}\, \frac{\partial\, 2\sigma_{11}^2 - 2\sigma_{11}\sigma_{22} - 2\sigma_{11}\sigma_{33}}{\partial \sigma_{11}}$$

$$= \frac{d\lambda}{6}\left(4\sigma_{11} - 2\sigma_{22} - 2\sigma_{33}\right) = d\lambda\left(\frac{2}{3}\sigma_{11} - \frac{1}{3}\sigma_{22} - \frac{1}{3}\sigma_{33}\right)$$

$$= d\lambda\, \frac{2}{3}\sigma_{11}$$

$$d\varepsilon_{22}^p = d\lambda\left(-\frac{1}{3}\sigma_{11}\right) \qquad d\varepsilon_{12}^p = 0$$

$$d\varepsilon_{33}^p = d\lambda\left(-\frac{1}{3}\sigma_{11}\right) \qquad d\varepsilon_{13}^p = 0$$

$$d\varepsilon_{23}^p = 0$$

$$d\bar{\varepsilon}^p = d\lambda\left\{\left[\frac{1}{4}\left(\frac{2}{3}\sigma_{11}\right)^2\right]^{\frac{1}{2}} + \left[\frac{1}{4}\left(-\frac{\sigma_{11}}{3}\right)^2\right]^{\frac{1}{2}} + \left[\frac{1}{4}\left(-\frac{\sigma_{11}}{3}\right)^2\right]^{\frac{1}{2}}\right\}$$

$$= d\lambda\, \frac{2}{3}\sigma_{11} = d\varepsilon_{11}^p = -2\, d\varepsilon_{22}^p = -2\, d\varepsilon_{33}^p$$

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For pure shear $\sigma_{12} = \sigma_{21}$

$$d\varepsilon^p_{12} = d\lambda \frac{\partial f}{\partial \sigma_{12}} = d\lambda \frac{\partial\left(\frac{1}{2}\sigma_{12}^2 + \frac{1}{2}\sigma_{21}^2\right)}{\partial \sigma_{12}} = d\lambda\, \sigma_{12}$$

(tensorial)

$$d\varepsilon^p_{11} = d\varepsilon^p_{22} = d\varepsilon^p_{33} = 0$$

$$d\varepsilon^p_{23} = d\varepsilon^p_{13} = 0$$

$$d\bar{\varepsilon}^p = \frac{1}{2}\left[\left(d\varepsilon^p_{12}\right)^2\right]^{\frac{1}{2}} + \frac{1}{2}\left[\left(d\varepsilon^p_{21}\right)^2\right]^{\frac{1}{2}} = d\varepsilon^p_{12}$$

Let us return to

$$d\varepsilon^p_{ij} = d\lambda\, S_{ij}'$$

$$d\varepsilon^p_{ij}\, d\varepsilon^p_{ij} = (d\lambda)^2 S_{ij}' S_{ij}'$$

$$4(d\bar{\varepsilon}^p)^2 = (d\lambda)^2 \frac{2}{3}\bar{\sigma}^2$$

$$d\lambda = \sqrt{6}\,\frac{d\bar{\varepsilon}^p}{\bar{\sigma}} = \sqrt{6}\,\frac{d\bar{\sigma}}{\bar{\sigma} h'} \qquad h' \neq 0$$

$$d\varepsilon^p_{ij} = \sqrt{6}\,\frac{S_{ij}'}{\bar{\sigma}}\, d\varepsilon^p$$

$$d\varepsilon^p_{ij} = \sqrt{6}\,\frac{S_{ij}'}{\bar{\sigma} h'}\, d\bar{\sigma} \qquad h' \neq 0$$

Flow Rule

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So, we have a plasticity model.

But the model is isotropic: the stress has the same effect regardless of the material direction.

How could we treat an anisotropic material?

Let us denote the isotropic stress space as IPE: Isotropic Plasticity Equivalent.
And let us transform the anisotropic stress space into IPE by a particular mapping matrix:

$$S = \mathcal{L}\,\Sigma$$

$$\mathcal{L} = C \begin{bmatrix} 1 & \beta_1 & \beta_2 & & & \\ \beta_1 & \alpha_1 & \beta_3 & & & \\ \beta_2 & \beta_3 & \alpha_2 & & & \\ & & & \gamma_1 & & \\ & & & & \gamma_4 & \\ & & & & & \gamma_3 \end{bmatrix}$$

What is then the relationship of the strain in IPE-space E^p and strain in the anisotropic space ε^p?

Plastic Work Density $\Sigma^T d\varepsilon^p = S^T dE^p$

$$\Rightarrow (\mathcal{L}\Sigma)^T dE^p = \Sigma^T \mathcal{L}^T dE^p = \Sigma^T d\varepsilon^p$$

$$\Rightarrow d\varepsilon^p = \mathcal{L}^T dE^p$$

Failure Criteria

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In stress space:

$$f(\sigma_{ij}) = 0$$

In strain space:

$$g(\varepsilon_{ij}) = 0$$

Size Effect on Strength

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Element Failure Probability
= cumulative distribution function (cdf) of stregth for an Element

pdf = probability density function

$$P_1(\sigma)$$

Element Survival Probability

$$1 - P_1(\sigma)$$

Chain survival probability

$$1 - P_f = (1 - P_1)^N \quad \Big| \ln \quad \Rightarrow \ln(1 - P_f) = N \ln(1 - P_1)$$

MacLaurin Series $f(x) = \sum_{n=0}^{\infty} f^n(0)\, x^n$

$$\frac{d}{dp} \ln(1-p) = \frac{d\,1-p}{dp} \frac{d \ln(1-p)}{d(1-p)} = -\frac{1}{1-p} = \frac{1}{p-1}$$

$$\ln(1 - P_1) = \ln(1) - P_1 + \dots \approx -P_1$$

$$\ln(1 - P_f) \approx -NP_1$$

Cumulative distriburtion function for the strength of the chain

$$P_f = 1 - e^{-NP_1(\sigma)}$$

$$N \equiv \frac{V}{V_1} \qquad \frac{P_1(\sigma)}{V_1} \equiv c(\sigma) \quad \Rightarrow \boxed{P_f(\sigma, V) = 1 - e^{-c(\sigma)V}}$$

Weibull 1939:

$$c(\sigma) \cong \frac{1}{V_0} \left\langle \frac{\sigma - \sigma_1}{\sigma_0} \right\rangle^m$$

$$\langle \text{abs}\, x \rangle = x$$

$$\langle -\text{abs}\, x \rangle = 0$$

$$\boxed{P_f(\sigma, V) = 1 - e^{-\frac{V}{V_0}\left\langle \frac{\sigma - \sigma_1}{\sigma_0} \right\rangle^m}}$$

$$\rightarrow \quad 1 - e^{-\frac{V}{V_0}\left\langle \frac{\sigma}{\sigma_0} \right\rangle^m} \quad \text{for } \sigma_1 = 0$$

What is the pdf of strength?

$$\frac{d}{d\sigma}P_f = \frac{d\,\frac{\sigma-\sigma_1}{\sigma_0}}{d\sigma}\;\frac{d\left(\frac{\sigma-\sigma_1}{\sigma_0}\right)^m}{d\,\frac{\sigma-\sigma_1}{\sigma_0}}\;\frac{d}{d\left(\frac{\sigma-\sigma_1}{\sigma_0}\right)^m}P_f = \frac{1}{\sigma_0}\,m\left(\frac{\sigma-\sigma_1}{\sigma_0}\right)^{m-1}\frac{V}{V_0}\,e^{-\frac{V}{V_0}\langle\,\rangle^m}$$

$$= \frac{m\,(\sigma-\sigma_1)^{m-1}}{\sigma_0^m}\,\frac{V}{V_0}\,e^{-\frac{V}{V_0}\left\langle\frac{\sigma-\sigma_1}{\sigma_0}\right\rangle^m}$$

$$\equiv p\text{-}f$$

What is the mean value of strength?

$$\bar{\sigma} = \int_{\sigma_1}^{\infty}\sigma\,p\text{-}f\,d\sigma = \int_0^1 \sigma\,dP_f$$

$$\frac{dP_f}{d\sigma} = p\text{-}f \;\Rightarrow\; p\text{-}f\,d\sigma = dP_f$$

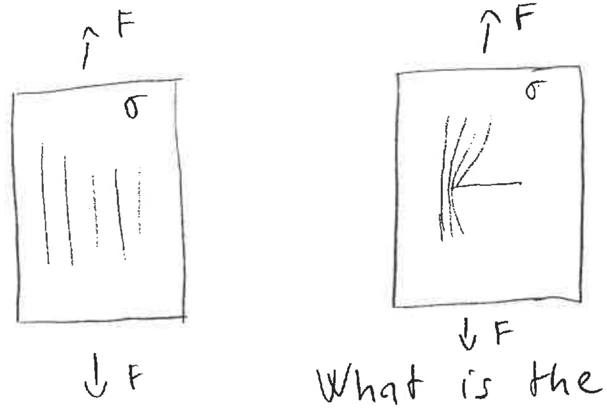
What is the median value of strength?

$$P_f(\sigma, V) = 0{,}5 \;\Rightarrow\; e^{-\frac{V}{V_0}\left\langle\frac{\sigma-\sigma_1}{\sigma_0}\right\rangle^m} = 0{,}5$$

$$\frac{V}{V_0}\left(\frac{\sigma-\sigma_1}{\sigma_0}\right)^m = \ln 2 \;\Rightarrow\; \sigma_{0,5} = \sigma_0\left(\frac{V_0}{V}\ln 2\right)^{\frac{1}{m}} + \sigma_1$$

What is wrong?

Stress-concentrating effect of a flaw



What is the local stress at the tip of a sharp crack?

Ideally there is an infinite local stress!

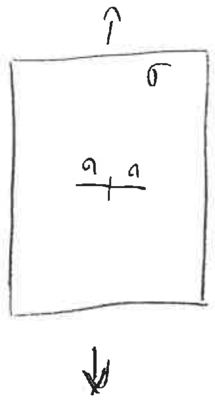
But what is a sharp crack?

There must be a curvature in the molecular or atomic scale → a very high local stress anyway!

– Any failure criterion, formulated in the stress space, is hardly able to reflect reality!

Energy Criterion by Griffith (1921):

A crack may grow if the crack growth releases at least as much potential energy as is needed for forming of new crack surface



$$\frac{d\Pi}{da} + \frac{dW_s}{da} \le 0$$

$$\frac{dW_s}{da} \le \frac{-d\Pi}{da} = \left|\frac{d\Pi}{da}\right|$$

$$\frac{dW_s}{da} = \frac{d\, 4ab\mathcal{G}}{da} = 4b\mathcal{G}$$

$$\mathcal{G} \equiv \text{Specific Fracture Energy} = \frac{dW_s}{dA}$$

Energy Release Rate $-\frac{d\Pi}{dA} = -\frac{d\Pi}{4b\,da}$

Critical Enegy Release Rate

$$\left(-\frac{d\Pi}{dA}\right)_c = \mathcal{G}$$

Solution for the crack effect on the Potential Energy by Inglis (1913):

$$\Pi = \Pi_0 - \frac{\pi \sigma^2 a^2 b}{Q}$$

for a large plate w. central through-crack

$$-\frac{d\Pi}{da} = \frac{2\pi\sigma^2 ab}{Q}$$

$$-\frac{d\Pi}{dA} = \frac{-d\Pi}{4b\,da} = \frac{\pi\sigma^2 a}{2Q}$$

$$\left(-\frac{d\Pi}{dA}\right)_c = \mathcal{G} = \frac{\pi\sigma_c^2 a}{2Q}$$

$$\Rightarrow \sigma_c = \sqrt{\frac{2\mathcal{G}Q}{\pi a}}$$

LEFM

solution for critical stress

$$K = \sigma\sqrt{\pi a}$$

$$K_c = \sigma_c\sqrt{\pi a} = \sqrt{2\mathcal{G}Q}$$

Solution for the crack effect on the Potential Energy by Inglis (1913): 54

$$\Pi = \Pi_0 - \frac{\pi \sigma^2 a^2 b}{Q}$$

for a large plate w. central through-crack

$$-\frac{d\Pi}{da} = \frac{2\pi\sigma^2 a b}{Q}$$

$$-\frac{d\Pi}{dA} = \frac{-d\Pi}{4b\,da} = \frac{\pi\sigma^2 a}{2Q}$$

$$\left(-\frac{d\Pi}{dA}\right)_c = G = \frac{\pi\sigma_c^2 a}{2Q}$$

$$\Rightarrow \sigma_c = \sqrt{\frac{2GQ}{\pi a}}$$

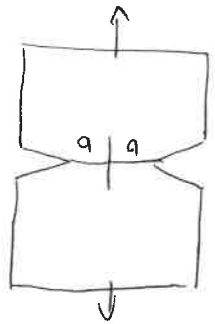
LEFM solution for critical stress

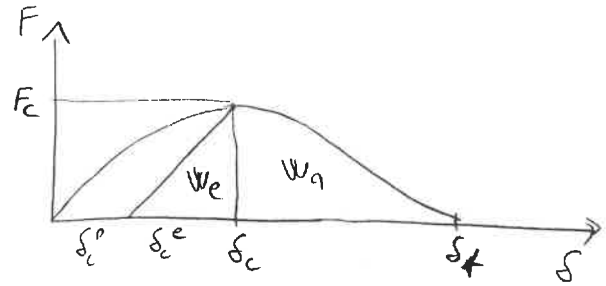
How do we determine G? 55

1° Use the critical stress Eq.

$$\sigma_c = \sqrt{\frac{2GQ}{\pi a}\beta} \Rightarrow G = \frac{\pi a \sigma_c^2}{2Q\beta}$$

2° Arrange a stable tensile experiment, and measure the work needed to separate the specimen faces:





$$W_e = \frac{1}{2} F_c \delta_c^e = \frac{1}{2} k (\delta_c^e)^2$$

$$W_a = \int_{\delta_c}^{\delta_f} F\,d\delta$$

$$W = \int_0^{\delta_f} F\,d\delta$$

$$\Delta W_s = W_e + W_a$$

$$G = \frac{\Delta W_s}{\Delta A_s} = \frac{W_e + W_a}{4ab}$$

$$W_p = W - W_e - W_a$$

Method 1° based on the Elastic Inglis solution

- Plastic yielding may change geometry
- Fracture process may take place within a zone not confined to the (pointwise) tip of the crack

LEFM

=> Method valid if

$a \gg$ plastic yielding zone size

$a \gg$ process zone size

2° Measures true Fracture Energy

- How does it apply to the LEFM Equation

$$\sigma_c = \sqrt{\frac{2\theta Q}{\pi a \beta}}$$

LEFM size effect:

$$\sigma_c \propto D^{-\frac{1}{2}}$$

How about a material w. plastic yielding at σ_{pe} ?

Let us try a scaling parameter

$$B = \frac{\sigma_{pe}^2}{\sigma_{c\_LEFM}^2} \equiv \text{Brittleness number} \equiv \frac{\pi a \beta \sigma_{pe}^2}{2\theta Q}$$

w. strength scaling

$$\sigma_c = \frac{\sigma_{pe}}{\sqrt{1+B}} = \frac{1}{\sqrt{\frac{1}{\sigma_{pe}^2} + \frac{1}{\sigma_{c\_LEFM}^2}}} = \frac{\sigma_{pe}}{\sqrt{1 + \frac{\pi a \beta \sigma_{pe}^2}{2\theta Q}}}$$

$$\equiv \frac{\sigma_{pe}}{\sqrt{1 + \frac{D}{l_{ch}}}} \qquad D \equiv \frac{\pi a \beta}{2}$$

$$l_{ch} \equiv \frac{\theta Q}{\sigma_{pe}^2}$$

Characteristic Material Length