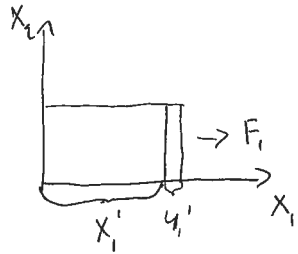


Forest Products Mechanics - Mechanics of Materials! (1)

Normal strain and stress



For small deformations

$$\epsilon_1 = \frac{u_1'}{x_1'} \quad \text{with} \quad u_1' \ll x_1'$$

$$\epsilon_{11} \equiv \lim_{\Delta x_1 \rightarrow 0} \frac{\Delta u_1}{\Delta x_1} = \frac{\partial u_1}{\partial x_1}$$

$$\epsilon_{11} = \lim_{\Delta x_1 \rightarrow 0} \frac{\Delta u_1}{\Delta x_1} = \frac{\partial u_1}{\partial x_1}$$

$$u_1' = \int_0^{x_1'} \frac{\partial u_1}{\partial x_1} dx_1 = \int_0^{x_1'} \epsilon_{11} dx_1$$

For ϵ_{11} varies

$$\Rightarrow \epsilon_{11} = \frac{u_1'}{x_1'}$$

Large Deformations?

Accumulated Average Strain

Boundary Displacement = relative change in length

$$\epsilon = \frac{dL}{L} = \frac{dx}{x}$$

$$\bar{\epsilon} = \int_x^{x+u} \frac{dx}{x} = \ln x \Big|_x^{x+u} = \ln(x+u) - \ln x = \ln \frac{x+u}{x} = \ln \left(1 + \frac{u}{x}\right)$$

Logarithmic Strain $\ln \left(1 + \frac{u}{x}\right)$

Engineering Strain $\frac{u}{x}$

Normal Strain Components

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2}$$

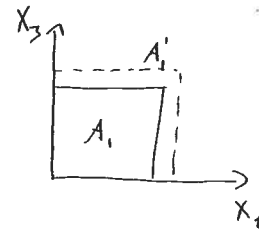
$$\epsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

Stress: Internal Force

(2)

At constant strain, displacement depends on length.

Is there some kind of relation between specimen size and Force needed to induce the strain?



$$F_1' = F_1 + \frac{\partial F_1}{\partial A_1} dA_1 = F_1 + \sigma_{11} dA_1 = F_1 + \sigma_{11} (A_1' - A_1)$$

Normal Stress Components

$$\sigma_{11} = \frac{\partial F_1}{\partial A_1}$$

$$\sigma_{22} = \frac{\partial F_2}{\partial A_2}$$

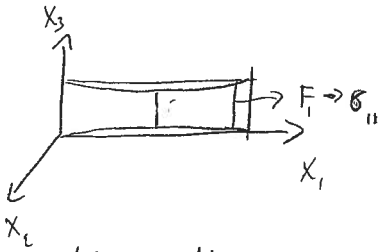
$$\sigma_{33} = \frac{\partial F_3}{\partial A_3}$$

Body: Boundary Force and Displacement - Measurable!

Material: Stress and Strain

Stress-Strain Relations (Normal Stresses and Strains) ③

$$\epsilon_i = \text{function}(\sigma_i) \quad ?$$



$$\epsilon_1 = \text{function}(\sigma_1, \sigma_2, \sigma_3)$$

$$\epsilon_2 = \text{function}(\sigma_1, \sigma_2, \sigma_3)$$

$$\epsilon_i = \text{function}(\sigma_j) \quad i, j \in \{1, 2, 3\}$$

Linear theory:

$$\epsilon_1 = S_{11}\sigma_1 + S_{12}\sigma_2 + S_{13}\sigma_3$$

$$\epsilon_2 = S_{21}\sigma_1 + S_{22}\sigma_2 + S_{23}\sigma_3$$

$$\epsilon_3 = S_{31}\sigma_1 + S_{32}\sigma_2 + S_{33}\sigma_3$$

$$\begin{aligned} \epsilon_i &= \sum_j S_{ij} \sigma_j = S_{ij} \sigma_j \\ &= S_{ij} \sigma_j \end{aligned}$$

In Matrix Form:
Compliance Matrix

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}$$

↑
strain vector components

stress vector components

$$\sigma_1 = Q_{11}\epsilon_1 + Q_{12}\epsilon_2 + Q_{13}\epsilon_3$$

$$\sigma_2 = Q_{21}\epsilon_1 + \dots$$

Stiffness Matrix

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix}$$

$$\begin{aligned} \sigma_k &= \sum_e Q_{ke} \epsilon_e = Q_{ke} \epsilon_e \\ &= Q_k^e \epsilon_e \end{aligned}$$

$$\begin{pmatrix} \epsilon \\ \vdots \\ \epsilon_i \\ \vdots \\ \epsilon_j \end{pmatrix} = \begin{pmatrix} S \\ \vdots \\ S_{ij} \\ \vdots \\ S \end{pmatrix} \begin{pmatrix} \sigma \\ \vdots \\ \sigma_j \\ \vdots \\ \sigma \end{pmatrix}$$

Volumetric Strain ④

$$\frac{V'}{V} = 1 + \frac{dV}{V} = \left(1 + \frac{u_1}{x_1}\right) \left(1 + \frac{u_2}{x_2}\right) \left(1 + \frac{u_3}{x_3}\right)$$

$$\ln\left(1 + \frac{dV}{V}\right) = \ln\left(1 + \frac{u_1}{x_1}\right) + \ln\left(1 + \frac{u_2}{x_2}\right) + \ln\left(1 + \frac{u_3}{x_3}\right)$$

$$= \epsilon_{11} + \epsilon_{22} + \epsilon_{33} \equiv \epsilon_v$$

Uniaxial Stress in direction j:

$$\epsilon_i = S_{ij} \sigma_j \quad (\text{no sum})$$

$$\epsilon_j = S_{jj} \sigma_j \quad (\text{no sum})$$

$$\sigma_j = E_j \epsilon_j \quad (\text{no sum})$$

Engineering Constants:

Young's Modulus

$$E_j = \frac{1}{S_{jj}} \quad (\text{no sum})$$

$$\epsilon_i = -\nu_{ij} \epsilon_j \quad (\text{ns})$$

$$S_{ij} \sigma_j = -\nu_{ij} \epsilon_j \quad (\text{ns})$$

$$S_{ij} E_j \epsilon_j = -\nu_{ij} \epsilon_j \quad (\text{ns}) \Rightarrow$$

$$S_{ij} = -\frac{\nu_{ij}}{E_j}$$

Poisson Ratio

Incompressible Material

$$\epsilon_v = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = 0$$

Isotropic Incompressible under Uniaxial stress in j

$$\epsilon_j = S_{jj} \sigma_j$$

$$\epsilon_k = -\nu_{kj} \epsilon_j = -\nu_{kj} S_{jj} \sigma_j$$

(no sums)

$$\epsilon_e = -\nu_{ej} \epsilon_j = -\nu_{ej} S_{jj} \sigma_j$$

$$\epsilon_j + \epsilon_k + \epsilon_e = 0 \Rightarrow (1 - \nu - \nu) S_{jj} \sigma_j = 0$$

$$\Rightarrow \nu = \frac{1}{2}$$

Isotropic Incompressible has Poisson Ratio of $\frac{1}{2}$

⑤

Total Differential of Displacement

⑥

$$du_i = \sum_j \frac{\partial u_i}{\partial x_j} dx_j = \frac{\partial u_i}{\partial x_j} dx_j = \epsilon_{ij} dx_j = \epsilon_{ji} dx_j$$

$$du_1 = \epsilon_{11} dx_1 + \epsilon_{12} dx_2 + \epsilon_{13} dx_3$$

$$du_2 = \epsilon_{21} dx_1 + \epsilon_{22} dx_2 + \epsilon_{23} dx_3$$

$$du_3 = \epsilon_{31} dx_1 + \epsilon_{32} dx_2 + \epsilon_{33} dx_3$$

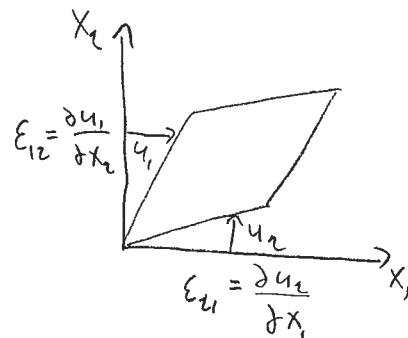
$$\begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$

Strain tensor - 9 components in 3-d space
Symmetric \Rightarrow 6 independent components

What are ϵ_{ij} , $i \neq j$?

Shear strains

$$\epsilon_{ij} = \frac{\partial u_i}{\partial x_j}$$



Why is the strain tensor symmetric?

Total Differential of force

(7)

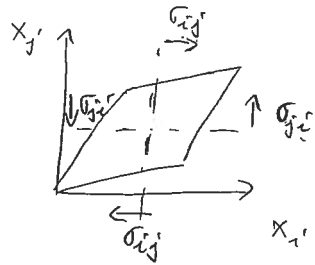
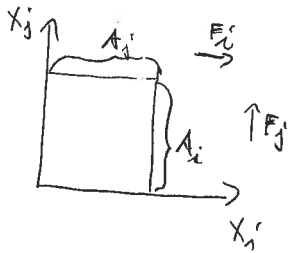
$$dF_i = \sum_j \frac{\partial F_i}{\partial A_j} dA_j = \sum \sigma_{ij} dA_j = \sigma_{ij} dA_j = \sigma_{ij} j dA_j$$

$$\begin{pmatrix} dF_1 \\ dF_2 \\ dF_3 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} dA_1 \\ dA_2 \\ dA_3 \end{pmatrix}$$

Stress tensor

Symmetric - 6 independent components

Shear Stresses $\sigma_{ij}, i \neq j$



Why is the stress tensor symmetric?

Stress-Strain Relations

(8)

$$\epsilon_{ij} = S_{ijkl} \sigma_{kl}$$

$$\sigma_{ij} = Q_{ijkl} \epsilon_{kl}$$

Compliance Matrix

$$\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{21} \\ \epsilon_{23} \\ \epsilon_{31} \\ \epsilon_{32} \end{pmatrix} = \begin{pmatrix} S_{1111} & S_{1122} & S_{1133} & S_{1112} & S_{1113} & S_{1121} & S_{1123} & S_{1131} & S_{1132} \\ S_{2211} & S_{2222} & S_{2233} & S_{2212} & S_{2213} & S_{2221} & S_{2223} & S_{2231} & S_{2232} \\ S_{3311} & S_{3322} & S_{3333} & S_{3312} & S_{3313} & S_{3321} & S_{3323} & S_{3331} & S_{3332} \\ S_{1211} & S_{1222} & S_{1233} & S_{1212} & S_{1213} & S_{1221} & S_{1223} & S_{1231} & S_{1232} \\ S_{1311} & S_{1322} & S_{1333} & S_{1312} & S_{1313} & S_{1321} & S_{1323} & S_{1331} & S_{1332} \\ S_{2111} & S_{2122} & S_{2133} & S_{2112} & S_{2113} & S_{2121} & S_{2123} & S_{2131} & S_{2132} \\ S_{2311} & S_{2322} & S_{2333} & S_{2312} & S_{2313} & S_{2321} & S_{2323} & S_{2331} & S_{2332} \\ S_{3111} & S_{3122} & S_{3133} & S_{3112} & S_{3113} & S_{3121} & S_{3123} & S_{3131} & S_{3132} \\ S_{3211} & S_{3222} & S_{3233} & S_{3212} & S_{3213} & S_{3221} & S_{3223} & S_{3231} & S_{3232} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{21} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{32} \end{pmatrix}$$

81 components

Stiffness Matrix

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \vdots \end{pmatrix} = \begin{pmatrix} Q_{1111} & Q_{1122} & Q_{1133} & Q_{1112} & Q_{1113} & Q_{1121} & Q_{1123} & Q_{1131} & Q_{1132} \\ Q_{2211} & Q_{2222} & Q_{2233} & Q_{2212} & Q_{2213} & Q_{2221} & Q_{2223} & Q_{2231} & Q_{2232} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \vdots \end{pmatrix}$$

81 components

Symmetry of Stiffness and Compliance (9)

Strain Energy Density $\hat{W} = \frac{1}{2} \epsilon_{ij} \sigma_{ij}'$

$$\frac{d\hat{W}}{d\epsilon_{ij}} = \sigma_{ij}'$$

$$\frac{d\hat{W}}{d\sigma_{ij}'} = \epsilon_{ij}$$

On the other hand:

$$\hat{W} = \frac{1}{2} (S_{ijke} \sigma_{ke}) \sigma_{ij}'$$

$$\frac{d\hat{W}}{d\sigma_{mn}} = \frac{1}{2} S_{mnlk} \sigma_{lk} + \frac{1}{2} S_{ijmn} \sigma_{ij}' = \frac{1}{2} (S_{mnij} + S_{ijmn}) \sigma_{ij}'$$

But this equals $\epsilon_{mn} = S_{mnij} \sigma_{ij}'$

$$\Rightarrow S_{mnij} = S_{ijmn}$$

Similarly for Stiffness:

$$\hat{W} = \frac{1}{2} \epsilon_{ij} (Q_{ijke} \epsilon_{ke})$$

$$\frac{d\hat{W}}{d\epsilon_{mn}} = \frac{1}{2} Q_{mnke} \epsilon_{ke} + \frac{1}{2} \epsilon_{ij} Q_{ijmn} = \frac{1}{2} (Q_{mnij} + Q_{ijmn}) \epsilon_{ij}$$

$$= \sigma_{mn} = Q_{mnij} \epsilon_{ij}'$$

$$\Rightarrow Q_{mnij} = Q_{ijmn} \Rightarrow 45 \text{ Independent Components}$$

Other symmetries of Compliance and Stiffness (10)

Due to symmetry of strain tensor:

$$Q_{ijke} = Q_{jike}$$

$$S_{ijke} = S_{jike}$$

$$Q_{ijke} = Q_{jiek}$$

$$S_{ijke} = S_{jiek}$$

Due to symmetry of stress tensor:

$$Q_{ijke} = Q_{jike}$$

$$S_{ijke} = S_{jiek}$$

\Rightarrow Matrices can be written as 6×6 square matrices.

$$\sigma_{ij} = \sigma_{ji}$$

Engineering Strain, $\epsilon_{ij} = \epsilon_{ij} + \epsilon_{ji} = 2\epsilon_{ij}'$

Symmetry $\rightarrow 21$ Independent Components

6x6 Compliance Matrix

(11)

ϵ_{11}	S_{1111}	S_{1122}	S_{1133}	$2 S_{1112}$	$2 S_{1113}$	$2 S_{1123}$	σ_{11}
ϵ_{22}	S_{2211}	S_{2222}	S_{2233}	$2 S_{2212}$	$2 S_{2213}$	$2 S_{2223}$	σ_{22}
ϵ_{33}	S_{3311}	S_{3322}	S_{3333}	$2 S_{3312}$	$2 S_{3313}$	$2 S_{3323}$	σ_{33}
$2 \epsilon_{12}$	$2 S_{1211}$	$2 S_{1222}$	$2 S_{1233}$	$4 S_{1212}$	$4 S_{1213}$	$4 S_{1223}$	σ_{12}
$2 \epsilon_{13}$	$2 S_{1311}$	$2 S_{1322}$	$2 S_{1333}$	$4 S_{1312}$	$4 S_{1313}$	$4 S_{1323}$	σ_{13}
$2 \epsilon_{23}$	$2 S_{2311}$	$2 S_{2322}$	$2 S_{2333}$	$4 S_{2312}$	$4 S_{2313}$	$4 S_{2323}$	σ_{23}

6x6 Stiffness Matrix \rightarrow 21 indep. compon

σ_{11}	Q_{1111}	Q_{1122}	Q_{1133}	Q_{1112}	Q_{1113}	Q_{1123}	ϵ_{11}
σ_{22}	Q_{2211}	Q_{2222}	Q_{2233}	Q_{2212}	Q_{2213}	Q_{2223}	ϵ_{22}
σ_{33}	Q_{3311}	Q_{3322}	Q_{3333}	Q_{3312}	Q_{3313}	Q_{3323}	ϵ_{33}
σ_{12}	Q_{1211}	Q_{1222}	Q_{1233}	Q_{1212}	Q_{1213}	Q_{1223}	$2 \epsilon_{12}$
σ_{13}	Q_{1311}	Q_{1322}	Q_{1333}	Q_{1312}	Q_{1313}	Q_{1323}	$2 \epsilon_{13}$
σ_{23}	Q_{2311}	Q_{2322}	Q_{2333}	Q_{2312}	Q_{2313}	Q_{2323}	$2 \epsilon_{23}$

Orthotropic Material, 0n-axis co-ordinates

(12)

- 2 perpendicular lines of reflection symmetry (2d)
- 3 perpendicular planes of reflection symmetry (3d)
- Shear stresses are not linked to normal strains
- Normal stresses are not linked to shear strains
- Shear strains are not linked to normal stresses
- Normal strains are not linked to shear stresses

Compliance Matrix - 9 independent components

ϵ_{11}	S_{1111}	S_{1122}	S_{1133}	σ_{11}
ϵ_{22}	S_{2211}	S_{2222}	S_{2233}	σ_{22}
ϵ_{33}	S_{3311}	S_{3322}	S_{3333}	σ_{33}
$2 \epsilon_{12}$		$4 S_{1212}$		σ_{12}
$2 \epsilon_{13}$			$4 S_{1313}$	σ_{13}
$2 \epsilon_{23}$				σ_{23}
			$4 S_{2323}$	

Stiffness Matrix - 9 independent components

σ_{11}	Q_{1111}	Q_{1122}	Q_{1133}	ϵ_{11}
σ_{22}	Q_{2211}	Q_{2222}	Q_{2233}	ϵ_{22}
σ_{33}	Q_{3311}	Q_{3322}	Q_{3333}	ϵ_{33}
σ_{12}			Q_{1212}	$2 \epsilon_{12}$
σ_{13}				$2 \epsilon_{13}$
σ_{23}				$2 \epsilon_{23}$
			Q_{1313}	
				Q_{2323}

Why are shear stresses not linked to normal strains, etc?

Any reflections wrt a symmetry plane changes the sign of one co-ordinate (or displacement)

\Rightarrow Normal strains and stresses invariant in symmetry transformations $(\epsilon_{ii})' = \frac{\partial u_i'}{\partial x_i'} =$

\Rightarrow Shear strains and stresses change sign in symmetry transformations $\frac{-\partial u_i'}{\partial x_j'} = \epsilon_{i'j'}$

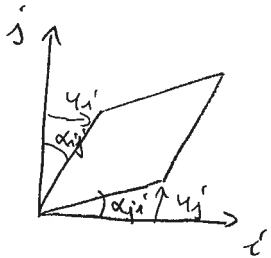
$(\epsilon_{i'j'})' = \frac{\partial u_{i'}}{\partial x_{j'}} = -\frac{\partial u_{i'}}{\partial x_{j'}} = -\epsilon_{i'j'}$

\Rightarrow Stiffnesses and compliances linking shear and normal strains/stresses should change sign in symmetry transformations. The only constant being equal to its negative is zero - this of course applies also to material constants!

Why are shear stresses σ_{ij} not linked to shear strains σ_{jk} ?

Reflection wrt symmetry plane (j,k) changes the sign of the co-ordinate (or displacement) i. Thus σ_{ij} changes sign, but σ_{jk} does not. The only material constant equal to its negative is again zero...

Measurement of Shear Strain



$$E_{ij} = E_{ji}$$

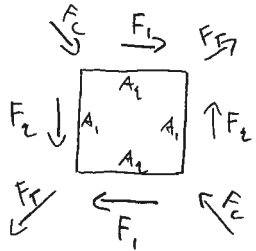
$$E_{ij} = \frac{\partial u_i}{\partial x_j} = -\tan \alpha_{ij}$$

$$E_{ji} = \frac{\partial u_j}{\partial x_i} = \tan \alpha_{ji}$$

$$\tan \alpha_{ij} = -\tan \alpha_{ji}$$

$$2E_{ij} = E_{ij} + E_{ji} = -\tan \alpha_{ij} + \tan \alpha_{ji} = 2 \tan |\alpha|$$

Measurement of Shear Stress



$$F_T = \cos 45^\circ F_1 + \cos(-45^\circ) F_2 = \frac{1}{\sqrt{2}} F_1 + \frac{1}{\sqrt{2}} F_2$$

$$A_T = \cos 45^\circ A_1 + \cos(-45^\circ) A_2 = \frac{1}{\sqrt{2}} A_1 + \frac{1}{\sqrt{2}} A_2$$

Set $A_1 = A_2$

$$\sigma_{12} = \sigma_{21} \Rightarrow \frac{\partial F_1}{\partial A_2} = \frac{\partial F_2}{\partial A_1} \Rightarrow F_1 = F_2$$

$$\sigma_T = \frac{F_T}{A_T} = \frac{F_1}{A_2} = \frac{F_2}{A_1} = \sigma_{12} = \sigma_{21} = \sigma_c$$

$$F_c = \cos(-45^\circ) F_1 + \cos 45^\circ F_2 = \sqrt{2} F_1 = \sqrt{2} F_2$$

$$A_c = \cos(-45^\circ) A_1 + \cos 45^\circ A_2 = \sqrt{2} A_1 = \sqrt{2} A_2$$

Shear Compliance in Orthotropy

$$S_{ijij} = \frac{E_{ij}}{\sigma_{ij}} = \frac{\tan \alpha}{\sigma_T} = \frac{\tan \alpha}{\sigma_c} \quad (\text{no sum})$$

$$4 S_{ijij} = \frac{4 \tan \alpha}{\sigma_T}$$

Shear Stiffness in Orthotropy

$$Q_{ijij} = \frac{\sigma_{ij}}{E_{ij}} = \frac{\sigma_T}{\tan \alpha} = \frac{\sigma_c}{\tan \alpha}$$

Uniaxial Stress Experiment in j

Engineering Constants

$$E_{ii} = S_{iijj} \sigma_{jj} \quad (\text{no sum})$$

$$1^\circ \quad i = j$$

$$2^\circ \quad i \neq j$$

$$E_j = \frac{\sigma_{jj}}{E_{jj}} \Rightarrow E_j = \frac{1}{S_{jjjj}} \quad (\text{no sum})$$

Poisson Ratio

$$E_{ii} = -\nu_{ij} E_{jj} = -\frac{\nu_{ij}}{E_j} \sigma_{jj}$$

Symmetry of Compliance

$$S_{iijj} = S_{jjii}$$

$$\Rightarrow \frac{\nu_{ij}}{E_j} = \frac{\nu_{ji}}{E_i}$$

$$\Rightarrow S_{iijj} = -\frac{\nu_{ij}}{E_j}$$

$$\Rightarrow \nu_{ij} = \frac{-S_{iijj}}{S_{jjjj}} \quad (\text{no sum})$$

Off-axis mechanical behavior of orthotropic materials

(15)

Off-axis stresses known – determine off-axis strains

- 1/ Stress transformation to on-axis
- 2/ Use on-axis compliance matrix to determine on-axis strains
- 3/ Negative transformation of strain to off-axis

Off-axis strains known – determine off-axis stresses

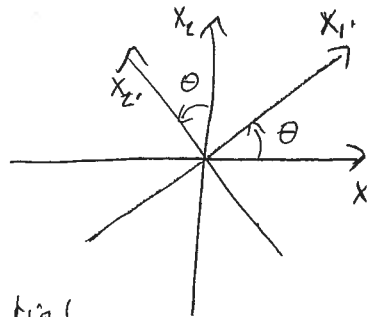
- 1/ Strain transformation to on-axis
- 2/ Use on-axis stiffness matrix to determine on-axis stresses
- 3/ Negative transformation of stress to off-axis

This cannot be done using the Engineering constants!

Co-ordinate transformation of stress in two Dimensions

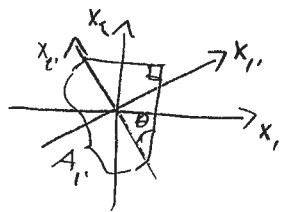
We are looking for a transformation matrix

$$\begin{pmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{1'2'} \end{pmatrix} = \begin{pmatrix} M_{\sigma} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}$$



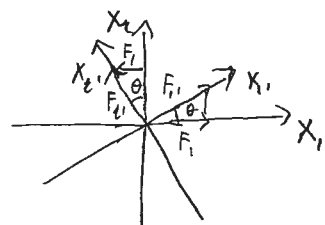
First, let us write the total differential of Force in \$x_1\$

$$\begin{aligned} dF_1 &= \frac{\partial F_1}{\partial A_1} dA_1 + \frac{\partial F_1}{\partial A_2} dA_2 \\ &= \sigma_{11} d(\cos\theta A_1) + \sigma_{12} d(\sin\theta A_1) \\ &= \sigma_{11} \cos\theta dA_1 + \sigma_{12} \sin\theta dA_1 \end{aligned}$$



Let us then divide \$dF_1\$ into components

$$\begin{aligned} dF_1 &= \cos\theta dF_{1'} + \sin\theta dF_{2'} \\ &= \cos\theta [\sigma_{1'1'} dA_{1'} + \sigma_{1'2'} dA_{2'}] \\ &\quad - \sin\theta [\sigma_{2'2'} dA_{2'} + \sigma_{2'1'} dA_{1'}] \end{aligned}$$



We find

$$\begin{aligned} \frac{dF_1}{dA_{1'}} &= \sigma_{11} \cos\theta + \sigma_{12} \sin\theta \\ \frac{dF_2}{dA_{1'}} &= \sigma_{1'1'} \cos\theta - \sigma_{2'1'} \sin\theta \end{aligned}$$

(16)

Let us then write the total differential of Force in \$x_2\$

$$dF_2 = \sigma_{21} dA_1 + \sigma_{22} dA_2 = \sigma_{21} \cos\theta dA_{1'} + \sigma_{22} \sin\theta dA_{1'}$$

And the components of \$dF_2\$

$$\begin{aligned} dF_2 &= \sin\theta dF_{1'} + \cos\theta dF_{2'} \\ &= \sin\theta [\sigma_{1'1'} dA_{1'} + \sigma_{1'2'} dA_{2'}] + \cos\theta [\sigma_{2'2'} dA_{2'} + \sigma_{2'1'} dA_{1'}] \end{aligned}$$

We find

$$\begin{aligned} \frac{dF_2}{dA_{1'}} &= \sigma_{21} \cos\theta + \sigma_{22} \sin\theta \\ \frac{dF_2}{dA_{1'}} &= \sigma_{1'1'} \sin\theta + \sigma_{2'1'} \cos\theta \end{aligned}$$

write
 $\cos\theta \equiv m$
 $\sin\theta \equiv n$

$$\begin{aligned} m\sigma_{11} + n\sigma_{12} &= m\sigma_{1'1'} - n\sigma_{2'1'} \\ m\sigma_{21} + n\sigma_{22} &= n\sigma_{1'1'} + m\sigma_{2'1'} \end{aligned}$$

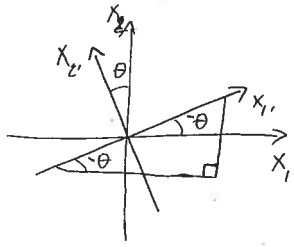
Two Equations,
 two unknowns:
 solve \$\sigma_{1'1'}\$
 and \$\sigma_{2'1'}\$

Result:

$$\begin{aligned} \sigma_{1'1'} &= m^2\sigma_{11} + n^2\sigma_{22} + 2mn\sigma_{12} \\ \sigma_{2'1'} = \sigma_{1'2'} &= -mn\sigma_{11} + mn\sigma_{22} + (m^2 - n^2)\sigma_{12} \end{aligned}$$

Now, we are missing $\sigma_{2'2'}$

(17)



Total differential of F_1

$$\begin{aligned} dF_1 &= \sigma_{11} dA_1 + \sigma_{12} dA_2 \\ &= \sigma_{11} d(\sin(\theta) A_{2'}) + \sigma_{12} d(\cos(\theta) A_{2'}) \\ &= -\sigma_{11} \sin \theta dA_{2'} + \sigma_{12} \cos \theta dA_{2'} \end{aligned}$$

Components

$$\begin{aligned} dF_1 &= \cos \theta dF_{1'} - \sin \theta dF_{2'} \\ &= \cos \theta [\sigma_{1'1'} dA_{1'} + \sigma_{1'2'} dA_{2'}] - \sin \theta [\sigma_{2'2'} dA_{2'} + \sigma_{2'1'} dA_{1'}] \end{aligned}$$

$$\begin{aligned} \frac{dF_1}{dA_{2'}} &= -\sigma_{11} \sin \theta + \sigma_{12} \cos \theta \\ \frac{dF_1}{dA_{2'}} &= \sigma_{1'2'} \cos \theta - \sigma_{2'2'} \sin \theta \end{aligned}$$

$$\begin{aligned} -n\sigma_{11} + m\sigma_{12} &= m\sigma_{1'2'} - n\sigma_{2'2'} \\ \Rightarrow \sigma_{2'2'} &= \sigma_{11} - \frac{m}{n}\sigma_{12} + \frac{m}{n}\sigma_{1'2'} \end{aligned}$$

$$\begin{aligned} &= \sigma_{11} - \frac{m}{n}\sigma_{12} + \frac{m}{n} \left[-mn\sigma_{11} + mn\sigma_{22} + (m^2 - n^2)\sigma_{12} \right] \\ &= (1 - m^2)\sigma_{11} + m^2\sigma_{22} + \left(-\frac{m}{n} + \frac{m^2}{n} - mn\right)\sigma_{12} \\ &= n^2\sigma_{11} + m^2\sigma_{22} + \frac{m}{n}(1 - m^2 - n^2)\sigma_{12} \\ &= \frac{mn}{n}(-2n^2)\sigma_{12} \\ &= n^2\sigma_{11} + m^2\sigma_{22} - 2mn\sigma_{12} \end{aligned}$$

All components known:

$$\begin{pmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \end{pmatrix} = \begin{pmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}$$

So, we have made a coord-transformation of stress. Was that right?

(18)

Inverse transformation should recover the initial state:

$$T^{-1} T = \mathbb{1}$$

$$\begin{aligned} T^{-1} &: (m)^{-1} = m \\ & (n)^{-1} = -n \end{aligned}$$

$$T_{\sigma}^{-1} = \begin{pmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2 - n^2 \end{pmatrix}$$

$$\begin{aligned} T^{-1} T &= \begin{pmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2 - n^2 \end{pmatrix} \begin{pmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{pmatrix} = \\ &= \begin{pmatrix} (m^2 + n^2)^2 & 0 & 0 \\ 0 & (m^2 + n^2)^2 & 0 \\ 0 & 0 & (m^2 + n^2)^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \% \end{aligned}$$

(19)

$$\epsilon_{ij} = \frac{\partial u_i}{\partial x_j} \quad \epsilon_{i'j'} = \frac{\partial u_{i'}}{\partial x_{j'}}$$

Chain rule of partial derivatives:

$$d\epsilon_{ij} = \frac{\partial \epsilon_{ij}}{\partial x_k} dx_k$$

Small-Strain Theory: $\epsilon_{i'j'} = \frac{\partial u_{i'}}{\partial x_{j'}} = \frac{\partial u_{i'}}{\partial x_k} \frac{\partial x_k}{\partial x_{j'}}$

$$\epsilon_{i'1'} = \frac{\partial u_{i'}}{\partial x_1} \frac{\partial x_1}{\partial x_{1'}} + \frac{\partial u_{i'}}{\partial x_2} \frac{\partial x_2}{\partial x_{1'}}$$

$$\epsilon_{i'2'} = \frac{\partial u_{i'}}{\partial x_1} \frac{\partial x_1}{\partial x_{2'}} + \frac{\partial u_{i'}}{\partial x_2} \frac{\partial x_2}{\partial x_{2'}}$$

...

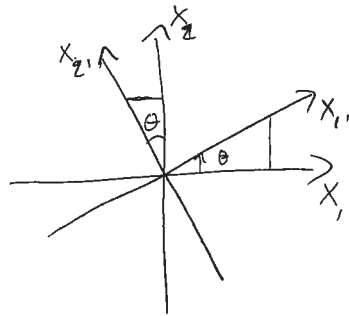
$$\epsilon_{1'1'} = \frac{\partial u_{1'}}{\partial x_1} m + \frac{\partial u_{1'}}{\partial x_2} n$$

$$= \left[\frac{\partial u_{1'}}{\partial x_1} \frac{\partial x_1}{\partial x_{1'}} + \frac{\partial u_{1'}}{\partial x_2} \frac{\partial x_2}{\partial x_{1'}} \right] m$$

$$+ \left[\frac{\partial u_{1'}}{\partial x_1} \frac{\partial x_1}{\partial x_{2'}} + \frac{\partial u_{1'}}{\partial x_2} \frac{\partial x_2}{\partial x_{2'}} \right] n$$

$$= \left[m\epsilon_{11} + n\epsilon_{21} \right] m + \left[m\epsilon_{12} + n\epsilon_{22} \right] n$$

$$= m^2\epsilon_{11} + n^2\epsilon_{22} + 2mn\epsilon_{12}$$



$$x_{1'} = \cos\theta x_1 + \sin\theta x_2$$

$$x_{2'} = -\sin\theta x_1 + \cos\theta x_2$$

$$x_1 = \cos\theta x_{1'} - \sin\theta x_{2'}$$

$$x_2 = \sin\theta x_{1'} + \cos\theta x_{2'}$$

Continued

(20)

$$\epsilon_{2'2'} = \frac{\partial u_{2'}}{\partial x_1} \frac{\partial x_1}{\partial x_{2'}} + \frac{\partial u_{2'}}{\partial x_2} \frac{\partial x_2}{\partial x_{2'}}$$

$$= \frac{\partial u_{2'}}{\partial x_1} (-n) + \frac{\partial u_{2'}}{\partial x_2} m$$

$$= \left[\frac{\partial u_{2'}}{\partial x_1} \frac{\partial x_1}{\partial x_1} + \frac{\partial u_{2'}}{\partial x_2} \frac{\partial x_2}{\partial x_1} \right] (-n) + \left[\frac{\partial u_{2'}}{\partial x_1} \frac{\partial x_1}{\partial x_2} + \frac{\partial u_{2'}}{\partial x_2} \frac{\partial x_2}{\partial x_2} \right] m$$

$$= \left[(-n)\epsilon_{11} + m\epsilon_{21} \right] (-n) + \left[-n\epsilon_{12} + m\epsilon_{22} \right] m$$

$$= n^2\epsilon_{11} + m^2\epsilon_{22} - 2mn\epsilon_{12}$$

$$\epsilon_{1'2'} = \frac{\partial u_{1'}}{\partial x_1} \frac{\partial x_1}{\partial x_{2'}} + \frac{\partial u_{1'}}{\partial x_2} \frac{\partial x_2}{\partial x_{2'}} = \left[\frac{\partial u_{1'}}{\partial x_1} \frac{\partial x_1}{\partial x_1} + \frac{\partial u_{1'}}{\partial x_2} \frac{\partial x_2}{\partial x_1} \right] (-n) + \left[\frac{\partial u_{1'}}{\partial x_1} \frac{\partial x_1}{\partial x_2} + \frac{\partial u_{1'}}{\partial x_2} \frac{\partial x_2}{\partial x_2} \right] m$$

$$= \left[m\epsilon_{11} + n\epsilon_{21} \right] (-n) + \left[m\epsilon_{12} + n\epsilon_{22} \right] m$$

$$= -mn\epsilon_{11} + mn\epsilon_{22} + (m^2 - n^2)\epsilon_{12}$$

$$\begin{pmatrix} \epsilon_{1'1'} \\ \epsilon_{2'2'} \\ \epsilon_{1'2'} \end{pmatrix} = \begin{pmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{pmatrix}$$

21

Co-ordinate transformation of Compliance

- 1/ Transform stress to on-axis
- 2/ Use on-axis compliance to compute strain
- 3/ Transform strain to off-axis
- 4/ Identify off-axis Compliance components – they are given as polynomials of on-axis compliance components!

Co-ordinate transformation of Stiffness

- 1/ Transform strain to on-axis
- 2/ Use on-axis stiffness to compute stress
- 3/ Transform stress to off-axis
- 4/ Identify off-axis Stiffness components – they are given as polynomials of on-axis stiffness components!

22

How do we determine Compliance Matrix experimentally?

Once we know Compliance Matrix, how do we get to know Stiffness Matrix?

If we know Stiffness Matrix, how do we get to know Compliance Matrix?

$$\left. \begin{aligned} \sigma &= Q \epsilon \\ \epsilon &= S \sigma \end{aligned} \right\} \begin{aligned} \epsilon &= SQ \epsilon \Rightarrow SQ = \mathbf{1} \Rightarrow Q = S^{-1} \\ \sigma &= QS \sigma \Rightarrow QS = \mathbf{1} \Rightarrow S = Q^{-1} \end{aligned}$$

How do we invert a matrix?

Gaussian Elimination:

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \quad \Bigg| \quad A^{-1}(\)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \quad \begin{aligned} ax + by &= x' \\ cx + dy &= y' \end{aligned} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\frac{cx + dy = y'}{b - \frac{ad}{c}} \quad | \cdot \left(-\frac{a}{c}\right)$$

$$\left(b - \frac{ad}{c}\right)y = x' - \frac{a}{c}y'$$

$$\begin{pmatrix} a & b \\ 0 & b - \frac{ad}{c} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -\frac{a}{c} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{aligned} ax + by &= x' \quad | \cdot \left(-\frac{b - \frac{ad}{c}}{b}\right) \\ \left(b - \frac{ad}{c}\right)y &= x' - \frac{a}{c}y' \end{aligned}$$

$$\frac{-\frac{a}{b} \left(b - \frac{ad}{c}\right) x = \left(1 - 1 + \frac{ad}{bc}\right) x' - \frac{a}{c} y'}{-\frac{a}{c} y'}$$

$$\begin{pmatrix} -\frac{a}{b}(b-\frac{ad}{c}) & 0 \\ 0 & b-\frac{ad}{c} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{ad}{bc} & -\frac{a}{c} \\ 1 & -\frac{a}{c} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \quad (23)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{ad}{bc}(-\frac{b}{a} + \frac{1}{b-\frac{ad}{c}}) & \frac{a}{c} \frac{b}{a} \frac{1}{b-\frac{ad}{c}} \\ \frac{1}{b-\frac{ad}{c}} & -\frac{a}{c} \frac{1}{b-\frac{ad}{c}} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{d}{c} \frac{1}{b-\frac{ad}{c}} & \frac{b}{c} \frac{1}{b-\frac{ad}{c}} \\ \frac{c}{bc-ad} & -\frac{a}{bc-ad} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-d}{bc-ad} & \frac{b}{bc-ad} \\ \frac{c}{bc-ad} & \frac{-a}{bc-ad} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Verify it:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \%$$

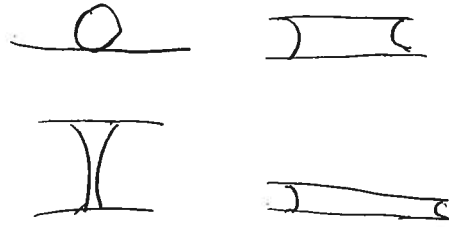
What if $(ad-bc) = 0$?

(24)

What is the Young's Modulus of Water?

(25)

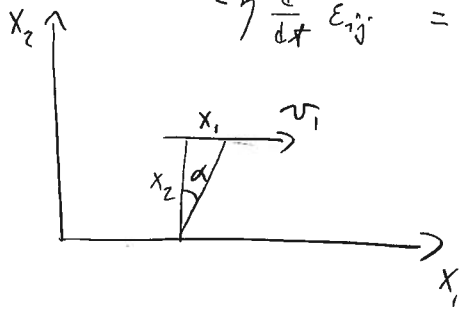
Negative in compression?



$$\sigma_{ij} = \text{function}(\dot{\epsilon}_{ke}) = \text{function}\left(\frac{d\epsilon_{ke}}{dt}\right)$$

Newtonian Viscosity for Isotropic Fluid:

$$\begin{aligned} \sigma_{ij} &= \eta \dot{\epsilon}_{ij} = \eta \frac{dv_i}{dx_j} = \eta \frac{d^2 u_i}{dt dx_j} = \eta \frac{d}{dt} \tan \alpha \\ &= \eta \frac{d}{dt} \epsilon_{ij} = \eta \dot{\epsilon}_{ij} \end{aligned}$$



Time-dependent Mechanical Behavior – Linear Viscoelasticity

(26)

$$\epsilon_{ij} = \sum_{jkl} C_{ijkl} \sigma_{kl}$$

$$\Rightarrow d\epsilon_{ij}(t) = C_{ijkl}(t-\tau) d\sigma_{kl}(\tau)$$

$$\epsilon_{ij}(t) = \int_{\sigma(\tau=-\infty)}^{\sigma(\tau=t)} C_{ijkl}(t-\tau) d\sigma_{kl}(\tau)$$

$$= \int_{-\infty}^t C_{ijkl}(t-\tau) \frac{d\sigma_{kl}}{d\tau} d\tau$$

$C_{ijkl}(t) \equiv$ Creep Compliance

$$\sigma_{ij} = R_{ijkl} \epsilon_{kl}$$

$$\Rightarrow d\sigma_{ij}(t) = R_{ijkl}(t-\tau) d\epsilon_{kl}(\tau)$$

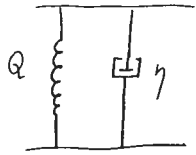
$$\sigma_{ij}(t) = \int_{-\infty}^t R_{ijkl}(t-\tau) \frac{d\epsilon_{kl}(\tau)}{d\tau} d\tau$$

$R_{ijkl}(t) \equiv$ Relaxation Modulus

What kind of a function is the Creep Compliance?

(27)

Voigt Element



$$\sigma = \epsilon Q + \dot{\epsilon} \eta$$

$$\frac{d\sigma}{dt} = 0 \Rightarrow \dot{\epsilon} Q + \ddot{\epsilon} \eta = 0$$

write $\dot{\epsilon} \equiv a \Rightarrow a Q + \dot{a} \eta = 0$

$$a Q = -\frac{da}{dt} \eta$$

$$-\frac{Q}{\eta} dt = \frac{da}{a} \quad | \int$$

$$-\frac{Q}{\eta} t = \ln a + C \quad | \text{exp}()$$

$$a = \dot{\epsilon} = C_2 e^{-\frac{Q}{\eta} t} \approx \frac{\sigma}{\eta} e^{-\frac{t}{\tau}}$$

$\tau \equiv$ Retardation Time

$$\frac{d\epsilon}{dt} = \frac{\sigma}{\eta} e^{-\frac{t}{\tau}}$$

$$d\epsilon = \frac{\sigma}{\eta} e^{-\frac{t}{\tau}} dt \quad | \int$$

$$\epsilon = -\frac{\sigma}{Q} e^{-\frac{t}{\tau}} + C_3$$

$$\epsilon(t=0) = -\frac{\sigma}{Q} + C_3 = 0 \Rightarrow C_3 = \frac{\sigma}{Q}$$

$$\epsilon = \frac{\sigma}{Q} (1 - e^{-\frac{t}{\tau}}) \Rightarrow \boxed{C(t) = C_\infty (1 - e^{-\frac{t}{\tau}})}$$

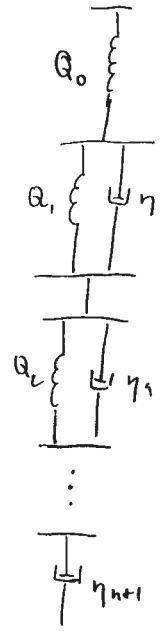
$\epsilon = C\sigma = \sigma \left[\frac{1}{Q} (1 - e^{-\frac{t}{\tau}}) \right]$

Voigt Elements in Series:

$$\epsilon = \sum_i (C_\infty)_i (1 - e^{-\frac{t}{\tau_i}}) \sigma$$

$$C(t) = \int_0^\infty (C_\infty)_i (1 - e^{-\frac{t}{\tau_i}}) di = \int_0^\infty C_\infty(\tau) (1 - e^{-\frac{t}{\tau}}) \frac{d\tau}{\tau} d\tau$$

What if two of the Voigt elements are degenerate? (28)



$$C(t) = C_0 + \int_0^\infty (C_\infty)_i (1 - e^{-\frac{t}{\tau_i}}) \frac{d\tau_i}{\tau_i} + \frac{t}{\eta}$$

$$= \boxed{C_0} + \int_{-\infty}^\infty \boxed{L(\tau)} (1 - e^{-\frac{t}{\tau}}) d(\ln \tau) + \frac{t}{\boxed{\eta}}$$

\uparrow Glassy Compliance
 \uparrow Retardation Spectrum
 \uparrow Steady-state Viscosity

Steady-state Compliance?
 Equilibrium
 Rubbery

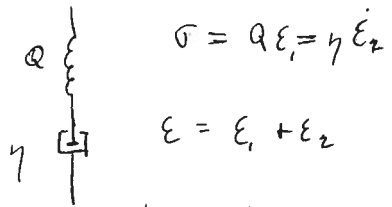
only if $\eta \rightarrow \infty$

$$C(\infty) = C_0 + \int_{-\infty}^\infty L(\tau) d(\ln \tau)$$

What kind of a function is the Relaxation Modulus?

(29)

Maxwell Element



$$\frac{d\epsilon}{dt} = \frac{d\epsilon_1}{dt} + \frac{d\epsilon_2}{dt} = 0$$

$$\frac{1}{Q} \frac{d\sigma}{dt} + \frac{\sigma}{\eta} = 0$$

$$\frac{d\sigma}{\sigma} = -\frac{Q}{\eta} dt$$

$$\ln \sigma = -\frac{Q}{\eta} t + C$$

$$\sigma = C_2 e^{-\frac{Q}{\eta} t} = R_0 \epsilon e^{-t/\tau}$$

$$R(t) = R_0 e^{-t/\tau}$$

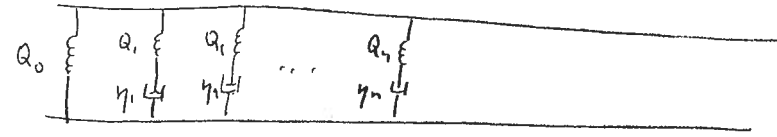
Maxwell elements in parallel:

$$\sigma = \sum_i \sigma_i = \sum_i R_i(t) \epsilon = \epsilon \sum_i (R_0)_i e^{-t/\tau_i}$$

$$R(t) = \sum_i (R_0)_i e^{-t/\tau_i} = \int_0^\infty (R_0)_i e^{-t/\tau_i} \frac{d_i}{d\tau} d\tau$$

What if some of the elements are degenerate?

(30)



$$R(t) = Q_0 + \sum_i (R_0)_i e^{-t/\tau_i}$$

$$= Q_0 + \int_0^\infty (R_0)_i e^{-t/\tau_i} \frac{d_i}{d\tau} d\tau$$

$$= \boxed{R_\infty} + \int_{-\infty}^\infty \boxed{H(\tau)} e^{-t/\tau} d(\ln \tau)$$

↑
Equilibrium Modulus

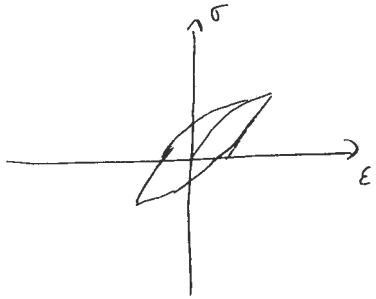
↑
Relaxation Spectrum

Classy Modulus:

$$R(t=0) = R_\infty + \int_{-\infty}^\infty H(\tau) d(\ln \tau) \equiv R_0$$

Cyclical Experiment

(31)

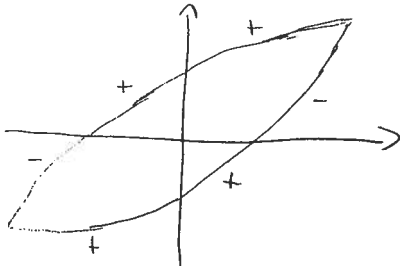


Strain Energy Density

$$W = \frac{1}{2} \sigma \epsilon = \frac{1}{2} Q \epsilon^2$$

$$\frac{dW}{d\epsilon} = \sigma$$

Where does the work (energy) go?



Stress - Strain - Time - Temperature - Moisture - relations

(32)

Crosslinked polymers:

Equilibrium Elasticity exist

$$\sigma = \epsilon \left[R_{\infty} + \sum_i (R_0)_i e^{-\frac{\epsilon}{\lambda_i}} \right]$$

Noncrosslinked: Liquid-like flow ($R_{\infty} = 0$)

Liquid-like flow

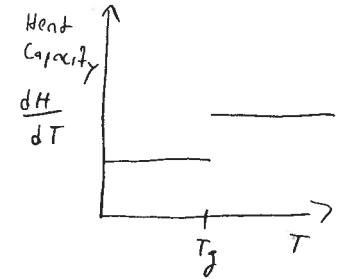
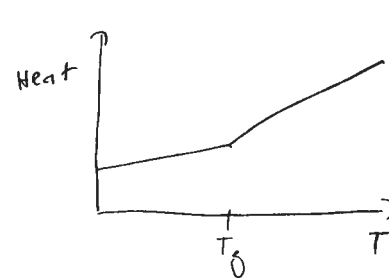
$$\epsilon = \sigma \left[\dots + \frac{1}{\eta} \right] \quad \eta < \infty$$

Amorphous Polymers:

- 1/ Large-deformation Equilibrium properties
- 2/ Small-deformation nonequilibrium properties (viscoelastic)
- 3/ Large-deformation time-dependent properties

Amorphous Polymers:

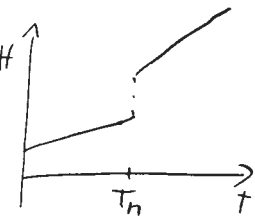
Glass Transition - second-order transition



Crystalline polymers:

Melting - first-order transition

First-order transition



1/ Large-deformation Equilibrium properties

(33)

From kinetic theory:

$$\sigma = \rho k T (\epsilon - \epsilon^{-2})$$

Eng. tensile stress
- " - strain

Shear Modulus $G = \rho k T$

How to approach this behavior?

- increase time
= reduce straining rate
- speed up relaxation (temperature, moisture, ...)

Time-Temperature Equivalency

(34)

- All characteristic times similarly affected by temperature change

Thermorheologically simple materials:

$$C(T, t) = C(T_0, t/a(T))$$

$$R(T, t) = R(T_0, t/b(T))$$

$$b(T) \approx a(T) \quad (?)$$

$$T > T_0 \Rightarrow t < t/a(T)$$

$$\Rightarrow a(T) < 1$$

$$T < T_0 \Rightarrow t > t/a(T)$$

$$\Rightarrow a(T) > 1$$

$$\boxed{\text{Reduced time} \equiv t/a(T)}$$

WLF:

$$\log a(T) = - \frac{c_1(T-T_0)}{c_2 + T - T_0} \approx \frac{-8,86(T-T_0)}{101,6 + T - T_0}$$

$$\log a = \frac{\ln a}{\ln 10}$$

$$10^{\log a} = a = 10^{- \frac{c_1(T-T_0)}{c_2 + T - T_0}}$$

Principal Stresses - stress Eigenvalues (35)

$$Ax = \lambda x \Rightarrow (A - \lambda I)x = 0$$

nontrivial solution $\Rightarrow |A - \lambda I| = 0$

{Characteristic secular determinant of the stress tensor = 0

$$(\sigma_{ij} - \sigma \delta_{ij}) n_j = 0$$

$$\begin{vmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma \end{vmatrix} = 0$$

$$(F) = (\Sigma) (A)$$

$$F = \Sigma A$$

$$\boxed{S^{-1} F} = S^{-1} \Sigma A$$

$$= \boxed{S^{-1} \Sigma S / S^{-1} A}$$

3rd degree polynomial \rightarrow 3 solutions

Hermitian matrix \rightarrow 3 real roots

3 stress Eigenvalues: $\sigma_1, \sigma_2, \sigma_3$

Stress tensor can be diagonalized by

Unitary Transformation \rightarrow Stresses in

$$S^{-1} \Sigma S = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

Normal Co-ordinates

The third-degree polynomial can be given in terms of stress Invariants (36)

$$\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0$$

$$I_1 = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$I_2 = \frac{1}{2} (\sigma_{ii} \sigma_{jj} - \sigma_{ij} \sigma_{ji})$$

$$= \sigma_{11} \sigma_{22} + \sigma_{22} \sigma_{33} + \sigma_{33} \sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2$$

$$I_3 = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix}$$

The stress Invariants can be given
in a co-ordinate system where the
stress tensor is diagonal
- in terms of principal stresses

(37)

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1$$

$$I_3 = \begin{vmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{vmatrix} = \sigma_1 \sigma_2 \sigma_3$$

How do we determine the direction
of a principal stress?

(38)

Once solving the Stress Eigenvalues, solve
the Eigenvalue Equation for Eigenvectors:

$$\sum \mathbf{x} = \sigma_i \mathbf{x} \quad \text{or}$$

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \sigma_i \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Where the values of x_1/x_2 and x_1/x_3
can be found.

Normalized Eigenvector: $x_1^2 + x_2^2 + x_3^2 = 1$

Elastoplasticity

(39)

$$\epsilon_{ij} = \epsilon_{ij}^e + \epsilon_{ij}^p$$

$$\sigma_{ij} = Q_{ijkl} \epsilon_{kl}^e = Q_{ijkl} (\epsilon_{kl} - \epsilon_{kl}^p)$$

Yield Function $f = f(\sigma_{ij})$

Yield condition $f=0 \Rightarrow$ yield surface

Isotropic Hardening -
yield surface expands

Kinematic Hardening -
yield surface translates

Ideal Plasticity -
yield surface stationary
in stress space

No stress state can exist outside of
the yield surface:

$f < 0$: Elastic range

$f = 0$: Plastic deformation may occur

Yield surface gradient:

(40)

- in co-ordinate space (3-d) $\nabla f = \bar{e}_i \frac{\partial}{\partial x_i} f$

- in stress space (9-d) $\nabla f = \bar{e}_{ij} \frac{\partial}{\partial \sigma_{ij}} f$

Gradient component: $\frac{\partial f}{\partial \sigma_{ij}}$

Plastic loading: $f = 0, \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} > 0$

$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} < 0 \Rightarrow$ Elastic Unloading

$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = 0 \Rightarrow$ Neutral Loading

Example of yield criterion,

(41)

formulated in 3-d principal stress space:

Von Mises
$$f = \frac{1}{6} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] - k^2 = 0$$

The von Mises criterion can be given in terms of Deviatoric Stresses

$$S_{ij} \equiv \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}$$

or

$$\begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} = \begin{pmatrix} \sigma_{11} - \frac{\sigma_{kk}}{3} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \frac{\sigma_{kk}}{3} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \frac{\sigma_{kk}}{3} \end{pmatrix}$$

The principal deviatoric stresses can be determined through the Eigenvalue Equation

(42)

$$Sx = sx \Rightarrow (S - sI)x = 0$$

$$\Rightarrow \det(S - sI) = 0$$

$$\begin{vmatrix} S_{11} - s & S_{12} & S_{13} \\ S_{21} & S_{22} - s & S_{23} \\ S_{31} & S_{32} & S_{33} - s \end{vmatrix} = 0$$

$$S^3 - J_1 S^2 - J_2 S - J_3 = 0$$

$$J_1 = S_{11} + S_{22} + S_{33} = S_{ii} = 0$$

$$J_2 = -(S_{11}S_{22} + S_{22}S_{33} + S_{33}S_{11} - S_{12}^2 - S_{13}^2 - S_{23}^2) = -\frac{1}{2} (S_{ii}S_{jj} - S_{ij}S_{ij})$$

$$J_3 = \det S = \frac{1}{6} \epsilon_{ijk} \epsilon_{pqr} S_{ip} S_{jq} S_{kr} = \frac{1}{2} S_{ij} S_{ij}$$

$$= \begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{vmatrix}$$

Levi-Civita
symbol
 ϵ_{ijk}

The deviatoric stress Invariant

(43)

$$\begin{aligned}
 J_2 &= \\
 &= (S_{11}S_{22} + S_{22}S_{33} + S_{33}S_{11} - S_{12}^2 - S_{13}^2 - S_{23}^2) \\
 &= \left[(\sigma_{11} - m)(\sigma_{22} - m) + (\sigma_{22} - m)(\sigma_{33} - m) + (\sigma_{33} - m)(\sigma_{11} - m) \right. \\
 &\quad \left. - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2 \right] \quad m = \frac{\sigma_{11}'}{3} \\
 &= - \left[\sigma_{11}\sigma_{22} - \sigma_{11}m - \sigma_{22}m + m^2 + \dots \right] \\
 &= - \left[\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - 2m(\sigma_{11} + \sigma_{22} + \sigma_{33}) + 3 \left(\frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} \right)^2 + \dots \right] \\
 &= - \left[\dots - \frac{2}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})^2 + \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})^2 + \dots \right] \\
 &= \frac{1}{6} \left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right] + \sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2
 \end{aligned}$$

change to a basis where Σ is diagonal

$$J_2 = \frac{1}{6} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

\Rightarrow Von Mises

Criterion becomes

$$\phi = J_2 - k^2 = 0 \quad !$$

Incremental Plasticity

(44)

Druoker postulate:

$$(\sigma_{ij} - \sigma_{ij}^o) d\epsilon_{ij}^p + d\sigma_{ij} d\epsilon_{ij}^p \geq 0$$

where σ_{ij} on the yield surface
 σ_{ij}^o on or inside the yield surface

$$\sigma_{ij}^o \text{ may equal } \sigma_{ij} \Rightarrow d\sigma_{ij} d\epsilon_{ij}^p \geq 0$$

$$\sigma_{ij} \text{ may be } \gg \sigma_{ij}^o \Rightarrow (\sigma_{ij} - \sigma_{ij}^o) d\epsilon_{ij}^p \geq 0$$

\Rightarrow yield surface is convex

$$\begin{cases}
 d\sigma_{ij} d\epsilon_{ij}^p \geq 0 \\
 d\sigma_{ij} \text{ limited by any tangent of the yield surface}
 \end{cases}$$

$\Rightarrow d\epsilon_{ij}^p$ is normal to the yield surface

$$d\epsilon_{ij}^p = d\lambda \frac{\partial \phi}{\partial \sigma_{ij}}$$

$$\begin{aligned}
 d\phi &= \frac{\partial \phi}{\partial \sigma_{ij}} d\sigma_{ij} = \frac{\partial \phi}{\partial \sigma_{ij}'} \frac{d\sigma_{ij}'}{d\lambda} d\lambda \\
 &= d\epsilon_{ij}^p \frac{d\sigma_{ij}'}{d\lambda}
 \end{aligned}$$

Invert von Mises

$$\sigma_{\text{eff}} = \sqrt{\frac{1}{2} (\sigma_{xx} - \sigma_{yy})^2 + \tau_{xy}^2} = \sqrt{\frac{1}{2} (\sigma_{xx}^2 - \sigma_{xx}\sigma_{yy} + \sigma_{yy}^2 + 2\tau_{xy}^2)}$$

$$\sigma_{\text{eff}} = \sqrt{\frac{1}{2} (\sigma_{xx}^2 - \sigma_{xx}\sigma_{yy} + \sigma_{yy}^2 + 2\tau_{xy}^2)}$$

Suppose a universal stress-strain-curve

$$\sigma = h(\epsilon^p) = h(\epsilon^p)$$

Define effective stress $\sigma_{\text{eff}} = \sqrt{\frac{1}{2} (\sigma_{xx}^2 - \sigma_{xx}\sigma_{yy} + \sigma_{yy}^2 + 2\tau_{xy}^2)}$

=> for uniaxial stress σ_{xx}

$$\sigma_{\text{eff}} = \sqrt{\frac{1}{2} (\sigma_{xx}^2 - \sigma_{xx}\sigma_{xx} + \sigma_{xx}^2 + 2\tau_{xy}^2)} = \sqrt{\frac{1}{2} (2\sigma_{xx}^2 + 2\tau_{xy}^2)} = \sqrt{\sigma_{xx}^2 + \tau_{xy}^2}$$

$$d\sigma_{\text{eff}} = \frac{d\sigma_{xx}}{\sqrt{2}} - \frac{\tau_{xy}}{\sqrt{2}} d\tau_{xy}$$

$$d\sigma_{\text{eff}} = \frac{d\sigma_{xx}}{\sqrt{2}} - \frac{\tau_{xy}}{\sqrt{2}} d\tau_{xy}$$

$$\frac{d\sigma_{\text{eff}}}{\sigma_{\text{eff}}} = \frac{d\sigma_{xx}}{\sigma_{xx}}$$

$$d\ln \sigma_{\text{eff}} = d\ln \sigma_{xx}$$

$$m = m(\sigma_{xx})$$

$$\frac{d\sigma_{\text{eff}}}{\sigma_{\text{eff}}} = \frac{d\sigma_{xx}}{\sigma_{xx}} - \frac{d\tau_{xy}}{\tau_{xy}}$$

$$d\sigma_{\text{eff}} = \frac{d\sigma_{xx}}{\sigma_{xx}} \sigma_{\text{eff}} + \frac{d\tau_{xy}}{\tau_{xy}} \sigma_{\text{eff}}$$

$$\sigma_{\text{eff}} = \sigma_{xx} (m) = \sigma_{xx} (m)$$

(45b)

(45)

Define effective plastic strain increment (46)

$$d\bar{\epsilon}^p \equiv \frac{1}{2} (d\epsilon_{ij}^p d\epsilon_{ij}^p)^{\frac{1}{2}} = \left(\frac{1}{4} d\epsilon_{ij}^p d\epsilon_{ij}^p \right)^{\frac{1}{2}}$$

For uniaxial stress σ_{11}

$$\begin{aligned} d\epsilon_{11}^p &= d\lambda \frac{\partial f}{\partial \sigma_{11}} = d\lambda \frac{1}{6} \frac{\partial \left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{33} - \sigma_{11})^2 \right]}{\partial \sigma_{11}} \\ &= d\lambda \frac{1}{6} \frac{\partial \left[2\sigma_{11}^2 - 2\sigma_{11}\sigma_{22} - 2\sigma_{11}\sigma_{33} \right]}{\partial \sigma_{11}} \end{aligned}$$

$$\begin{aligned} &= \frac{d\lambda}{6} (4\sigma_{11} - 2\sigma_{22} - 2\sigma_{33}) = d\lambda \left(\frac{2}{3}\sigma_{11} - \frac{1}{3}\sigma_{22} - \frac{1}{3}\sigma_{33} \right) \\ &= d\lambda \frac{2}{3}\sigma_{11} \end{aligned}$$

$$d\epsilon_{22}^p = d\lambda \left(-\frac{1}{3}\sigma_{11} \right) \quad d\epsilon_{12}^p = 0$$

$$d\epsilon_{33}^p = d\lambda \left(-\frac{1}{3}\sigma_{11} \right) \quad d\epsilon_{13}^p = 0$$

$$d\epsilon_{23}^p = 0$$

$$d\bar{\epsilon}^p = d\lambda \left\{ \left[\frac{1}{4} \left(\frac{2}{3}\sigma_{11} \right)^2 \right]^{\frac{1}{2}} + \left[\frac{1}{4} \left(-\frac{\sigma_{11}}{3} \right)^2 \right]^{\frac{1}{2}} + \left[\frac{1}{4} \left(-\frac{\sigma_{11}}{3} \right)^2 \right]^{\frac{1}{2}} \right\}$$

$$= d\lambda \frac{2}{3}\sigma_{11} = d\epsilon_{11}^p = -2d\epsilon_{22}^p = -2d\epsilon_{33}^p$$

For pure shear $\sigma_{12} = \sigma_{21}$ (47)

$$d\epsilon_{12}^p = d\lambda \frac{\partial f}{\partial \sigma_{12}} = d\lambda \frac{\partial \left(\frac{1}{2}\sigma_{12}^2 + \frac{1}{2}\sigma_{21}^2 \right)}{\partial \sigma_{12}} = d\lambda \sigma_{12}$$

(tensorial)

$$d\epsilon_{11}^p = d\epsilon_{22}^p = d\epsilon_{33}^p = 0$$

$$d\epsilon_{23}^p = d\epsilon_{13}^p = 0$$

$$d\bar{\epsilon}^p = \frac{1}{2} \left[(d\epsilon_{12}^p)^2 \right]^{\frac{1}{2}} + \frac{1}{2} \left[(d\epsilon_{21}^p)^2 \right]^{\frac{1}{2}} = d\bar{\epsilon}_{12}^p$$

Let us return to

$$d\epsilon_{ij}^p = d\lambda s_{ij}$$

$$d\epsilon_{ij}^p d\epsilon_{ij}^p = (d\lambda)^2 s_{ij} s_{ij}$$

$$4(d\bar{\epsilon}^p)^2 = (d\lambda)^2 \frac{2}{3} \bar{\sigma}^2$$

$$d\lambda = \sqrt{6} \frac{d\bar{\epsilon}^p}{\bar{\sigma}} = \sqrt{6} \frac{d\bar{\sigma}}{\bar{\sigma} h'} \quad h' \neq 0$$

$$d\epsilon_{ij}^p = \sqrt{6} \frac{s_{ij}}{\bar{\sigma}} d\bar{\epsilon}^p$$

$$d\epsilon_{ij}^p = \sqrt{6} \frac{s_{ij}}{\bar{\sigma} h'} d\bar{\sigma} \quad h' \neq 0 \quad \text{Flow Rule}$$

Size Effect on Strength

(50)

Element Failure Probability

= cumulative distribution function (cdf) of strength for an Element

$$P_i(\sigma)$$

pdf = probability density function

Element Survival Probability

$$1 - P_i(\sigma)$$

Chain survival probability

$$1 - P_f = (1 - P_i)^N \quad | \quad \ln \Rightarrow \ln(1 - P_f) = N \ln(1 - P_i)$$

MacLaurin Series

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0) x^n$$

$$\ln(1 - P_i) = \ln(1 - P_i) + \dots \approx -P_i$$

$$\ln(1 - P_f) \approx -NP_i$$

$$\left. \begin{aligned} \frac{d}{dP} \ln(1 - P) &= \\ \frac{d(1 - P)}{dP} \frac{d \ln(1 - P)}{d(1 - P)} &= \\ -\frac{1}{1 - P} &= \frac{1}{P - 1} \end{aligned} \right\}$$

Cumulative distribution function for the strength of the chain

$$P_f = 1 - e^{-NP_i(\sigma)}$$

$$N \equiv \frac{V}{V_i} \quad \frac{P_i(\sigma)}{V_i} \equiv c(\sigma) \Rightarrow P_f(\sigma, V) = 1 - e^{-c(\sigma)V}$$

Weibull 1939:

$$c(\sigma) \equiv \frac{1}{V_0} \left\langle \frac{\sigma - \sigma_i}{\sigma_0} \right\rangle^m$$

$$\langle \text{abs } x \rangle = x$$

$$\langle -\text{abs } x \rangle = 0$$

$$P_f(\sigma, V) = 1 - e^{-\frac{V}{V_0} \left\langle \frac{\sigma - \sigma_i}{\sigma_0} \right\rangle^m}$$

$$\rightarrow 1 - e^{-\frac{V}{V_0} \left\langle \frac{\sigma}{\sigma_0} \right\rangle^m} \quad \text{for } \sigma_i = 0$$

What is the pdf of strength?

(51)

$$\begin{aligned} \frac{d}{d\sigma} P_f &= \frac{d \frac{\sigma - \sigma_i}{\sigma_0}}{d\sigma} \frac{d \left(\frac{\sigma - \sigma_i}{\sigma_0} \right)^m}{d \left(\frac{\sigma - \sigma_i}{\sigma_0} \right)} \frac{d}{d \left(\frac{\sigma - \sigma_i}{\sigma_0} \right)^m} P_f = \frac{1}{\sigma_0} m \left(\frac{\sigma - \sigma_i}{\sigma_0} \right)^{m-1} \frac{V}{V_0} e^{-\frac{V}{V_0} \left\langle \frac{\sigma - \sigma_i}{\sigma_0} \right\rangle^m} \\ &= \frac{m \left(\frac{\sigma - \sigma_i}{\sigma_0} \right)^{m-1}}{\sigma_0^m} \frac{V}{V_0} e^{-\frac{V}{V_0} \left\langle \frac{\sigma - \sigma_i}{\sigma_0} \right\rangle^m} \\ &\equiv p-f \end{aligned}$$

What is the mean value of strength?

$$\begin{aligned} \bar{\sigma} &= \int_{\sigma_i}^{\infty} \sigma p-f d\sigma \\ &= \int_0^{\infty} \sigma dP_f \end{aligned} \quad \left/ \quad \begin{aligned} \frac{dP_f}{d\sigma} &= p-f \\ \Rightarrow p-f d\sigma &= dP_f \end{aligned} \right.$$

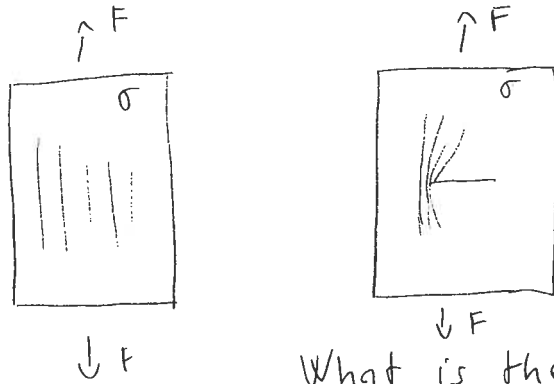
What is the median value of strength?

$$\begin{aligned} P_f(\sigma, V) = 0,5 &\Rightarrow e^{-\frac{V}{V_0} \left\langle \frac{\sigma - \sigma_i}{\sigma_0} \right\rangle^m} = 0,5 \\ \frac{V}{V_0} \left\langle \frac{\sigma - \sigma_i}{\sigma_0} \right\rangle^m &= \ln 2 \Rightarrow \sigma_{0,5} = \sigma_0 \left(\frac{V_0}{V} \ln 2 \right)^{\frac{1}{m}} + \sigma_i \end{aligned}$$

What is wrong?

Stress-concentrating effect of a flaw

(52)



What is the local stress at the tip of a sharp crack?

Ideally there is an infinite local stress!

But what is a sharp crack?

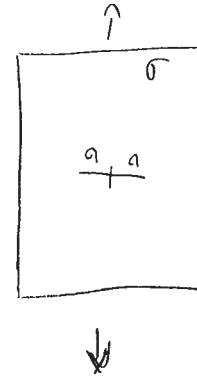
There must be a curvature in the molecular or atomic scale → a very high local stress anyway!

- Any failure criterion, formulated in the stress space, is hardly able to reflect reality!

Energy Criterion by Griffith (1921):

(53)

A crack may grow if the crack growth releases at least as much energy as is needed for forming of new crack surface ^{potential}



$$\frac{d\Pi}{da} + \frac{dW_s}{da} \leq 0$$

$$\frac{dW_s}{da} \leq -\frac{d\Pi}{da} = \left| \frac{d\Pi}{da} \right|$$

$$\frac{dW_s}{da} = \frac{d(4ab\theta)}{da} = 4b\theta$$

$$\begin{aligned} \theta &\equiv \text{Specific Fracture Energy} \\ &= \frac{dW_s}{dA} \end{aligned}$$

$$\text{Energy Release Rate } -\frac{d\Pi}{dA} = -\frac{d\Pi}{4b da}$$

Critical Energy Release Rate

$$\left(-\frac{d\Pi}{dA} \right)_c = \theta$$

Solution for the crack effect on the Potential Energy by Inglis (1913):

(54)

$$\Pi = \Pi_0 - \frac{\tilde{\pi} \sigma^2 a^2 b}{Q} \quad \text{for a large plate w. central through-crack}$$

$$-\frac{d\Pi}{da} = \frac{2\tilde{\pi} \sigma^2 a b}{Q}$$

$$-\frac{d\Pi}{dA} = \frac{-d\Pi}{4b da} = \frac{\tilde{\pi} \sigma^2 a}{2Q}$$

$$\left(-\frac{d\Pi}{dA}\right)_c = \Theta = \frac{\tilde{\pi} \sigma_c^2 a}{2Q}$$

$$\Rightarrow \sigma_c = \sqrt{\frac{2\Theta Q}{\tilde{\pi} a}}$$

LEFM

Solution for critical stress

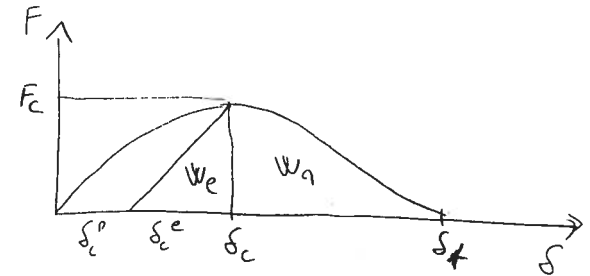
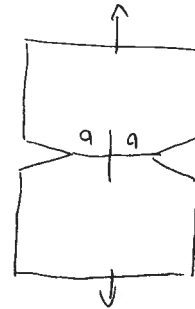
How do we determine G?

(55)

1° Use the critical stress Eq.

$$\sigma_c = \sqrt{\frac{2\Theta Q}{\tilde{\pi} a}} \Rightarrow \Theta = \frac{\tilde{\pi} a \sigma_c^2}{2Q\beta}$$

2° Arrange a stable tensile experiment, and measure the work needed to separate the specimen faces:



$$W_e = \frac{1}{2} F_c \delta_c^e = \frac{1}{2} k (\delta_c^e)^2$$

$$W_a = \int_{\delta_c}^{\delta_f} F d\delta$$

$$\Delta W_s = W_e + W_a$$

$$G = \frac{\Delta W_s}{\Delta A_s} = \frac{W_e + W_a}{4ab}$$

$$W = \int_0^{\delta_f} F d\delta$$

$$W_p = W - W_e - W_a$$

How do these two approaches differ?

(56)

Method 1^o based on the Elastic

Inglis solution - plastic yielding may change geometry

- Fracture process may take place within a zone not confined to the (pointwise) tip of the crack

LEFM

⇒ Method valid if $a \gg$ plastic yielding zone size
 $a \gg$ process zone size

2^o Measures true Fracture Energy

- How does it apply to the

LEFM Equation

$$\sigma_c = \sqrt{\frac{2\theta Q}{\pi a \beta}}$$

Fracture Mechanics Size Effect

(57)

LEFM size effect:

$$\sigma_c \propto D^{-1/2}$$

How about a material w. plastic yielding at σ_{pe} ?

Let us try a scaling parameter

$$B = \frac{\sigma_{pe}^2}{\sigma_{c,LEFM}^2} \equiv \text{Brittleness number} \\ \equiv \frac{\pi a \beta \sigma_{pe}^2}{2\theta Q}$$

w. strength scaling

$$\sigma_c = \frac{\sigma_{pe}}{\sqrt{1+B}} = \frac{1}{\sqrt{\frac{1}{\sigma_{pe}^2} + \frac{1}{\sigma_{c,LEFM}^2}}} = \frac{\sigma_{pe}}{\sqrt{1 + \frac{\pi a \beta \sigma_{pe}^2}{2\theta Q}}}$$

$$\equiv \frac{\sigma_{pe}}{\sqrt{1 + \frac{D}{l_{ch}}}}$$

$$D \equiv \frac{\pi a \beta}{2}$$

$$l_{ch} \equiv \frac{\theta Q}{\sigma_{pe}^2}$$

Characteristic Material Length