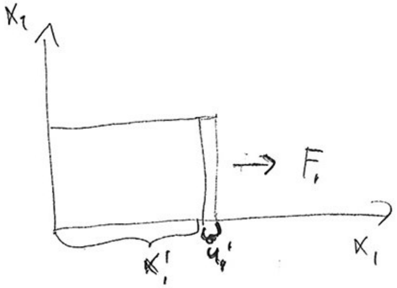
Normal strain and stress

3.9.06

①



$$\varepsilon_1 = \frac{u_1'}{x_1'} \quad ?$$

$$\varepsilon_1 = \lim_{\Delta x \to 0} \frac{\Delta u_1}{\Delta x_1} = \frac{\partial u_1}{\partial x_1}$$

$$u_1 = \int_0^{x_1'} \frac{\partial u_1}{\partial x_1} dx_1 = \int_0^{x_1'} \varepsilon_1 \, dx_1$$

~~Accumulated~~ Average strain?

$$\bar{\varepsilon}_1 = \int_{x_1'}^{x_1'+u_1} \frac{dx}{x} = \Big|_{x_1}^{x_1+u_1} \ln x = \ln(x_1'+u_1) - \ln x_1$$

$$\int_{x_1'}^{x_1'+u_1} \frac{dL}{L}$$

$$= \ln \frac{x_1'+u_1}{x_1'} = \ln\left(1+\frac{u_1}{x_1'}\right)$$

$$\varepsilon_1 = \frac{\partial u_1}{\partial x_1} \qquad \varepsilon_2 = \frac{\partial u_2}{\partial x_2} \qquad \varepsilon_3 = \frac{\partial u_3}{\partial x_3}$$

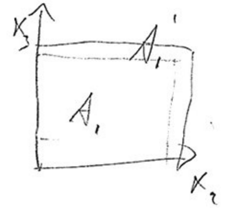
Normal strain components

So: at constant strain, displacement depends on length.

Is there some kind of relation between specimen size and force needed to induce the strain?

$$\sigma_1 = \frac{\partial F_1}{\partial A_1} \qquad \sigma_2 = \frac{\partial F_2}{\partial A_2} \qquad \sigma_3 = \frac{\partial F_3}{\partial A_3}$$

Normal stress components



$$F_{1i}' = F_{1i} + \frac{\partial F_{1i}}{\partial A_1} dA_1$$

$$= F_{1i} + \sigma_1 (A_1' - A_1)$$

Stress: Internal force

If σ_1 constant:

$$F_1 = \sigma_1 A_1 \Rightarrow \sigma_1 = \frac{F_{1i}}{A_1}$$

Body experiences Force and boundary displacement | Measurable

Material experiences stress and strain

Material elements ...

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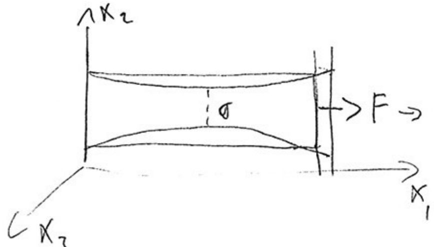
Stress-strain relations - Material behavior (?) ②

$$\varepsilon_1 = \text{function}(\sigma_1) \quad ?$$



$\to F \to \sigma$

$$\varepsilon_1 = \text{function}(\sigma_1, \sigma_2, \sigma_3)$$

$$\varepsilon_2 = \text{function}(- " -)$$

$$\varepsilon_i = \text{function}(\sigma_j), \quad i, j \in \{1,2,3\}$$

$$\varepsilon_1 = S_{11}\sigma_1 + S_{12}\sigma_2 + S_{13}\sigma_3$$

$$\varepsilon_2 = S_{21}\sigma_1 + S_{22}\sigma_2 + S_{23}\sigma_3$$

$$\varepsilon_3 = S_{31}\sigma_1 + S_{32}\sigma_2 + S_{33}\sigma_3$$

$$\varepsilon_i = \sum_j S_{ij}\sigma_j = S_i^{\ j}\sigma_j$$

$$= \boxed{\varepsilon_i = S_i^{\ j}\sigma_j}$$

Compliance matrix

$$\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13}\\ S_{21} & S_{22} & S_{23}\\ S_{31} & S_{32} & S_{33}\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}$$

$$\begin{pmatrix} \varepsilon \end{pmatrix}_i = \begin{pmatrix} \ \end{pmatrix}_i^{\ j} \begin{pmatrix} \sigma \end{pmatrix}_j$$

$$\sigma_1 = Q_{11}\varepsilon_1 + Q_{12}\varepsilon_2 + Q_{13}\varepsilon_3$$

$$\sigma_2 = Q_{21}\varepsilon_1 + \dots$$

$$\sigma_k = Q_k^{\ e}\varepsilon_e = \sum_e Q_{ke}\varepsilon_e$$

stiffness matrix

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13}\\ Q_{21} & Q_{22} & Q_{23}\\ Q_{31} & Q_{32} & Q_{33}\end{pmatrix}\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{pmatrix}$$

Matrices characteristic for the material

- linear (idealized)

Young's modulus = modulus of elasticity??

Examples - First Volumetric strain 3.9.06

Uniaxial stress in j-direction:

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$$\varepsilon_i = S_{ij}\,\sigma_j$$

$$\varepsilon_j = S_{jj}\,\sigma_j \quad \text{(no sum)} \quad = \frac{1}{E_j}\,\sigma_j$$

$$i \neq j: \quad \varepsilon_i = -\nu_{ij}\,\varepsilon_j = -\nu_{ij}\,S_{jj}\,\sigma_j \quad \text{(no sum)}$$

toisaalta $\varepsilon_i = S_{ij}\,\sigma_j$ (no sum)

$$\Rightarrow \quad S_{ij} = -\nu_{ij}\,S_{jj} = -\frac{\nu_{ij}}{E_j}$$

Isotropic ~~Homogeneous~~, Incompressible material:

Uniaxial stress in j

$$\boxed{\sum_i \varepsilon_i = 0} = \varepsilon_j + \varepsilon_k + \varepsilon_l = 0 \qquad j \neq k \neq l$$

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0 = \text{Volumetric strain} = 1-(1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3)$$

$$\varepsilon_j = S_{jj}\,\sigma_j$$

$$\varepsilon_k = -\nu_{kj}\,S_{jj}\,\sigma_j$$

$$\varepsilon_l = -\nu_{lj}\,S_{jj}\,\sigma_j$$

$$\nu_{kj} = \nu_{lj} = \nu \quad \text{(Isotropy)}$$

$$\varepsilon_j + \varepsilon_k + \varepsilon_l = (1-\nu-\nu)\,S_{jj}\,\sigma_j = 0$$

$$\Rightarrow \quad \nu = \frac{1}{2}$$

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Volumetric strain

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$$\varepsilon_v = (1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) - 1 \approx \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

Total differential of displacement

$$du_i = \frac{\partial u_i}{\partial x_j}\,dx_j = \sum_j \varepsilon_{ij}\,dx_j = \varepsilon_{ij}\,dx_j$$

$$du_1 = \varepsilon_{11}\,dx_1 + \varepsilon_{12}\,dx_2 + \varepsilon_{13}\,dx_3$$

$$du_2 = \varepsilon_{21}\,dx_1 + \varepsilon_{22}\,dx_2 + \varepsilon_{23}\,dx_3$$

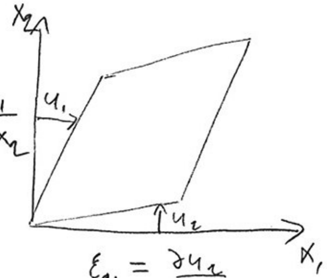
$$du_3 = \varepsilon_{31}\,dx_1 + \varepsilon_{32}\,dx_2 + \varepsilon_{23}\,dx_3$$

$$\varepsilon_{ij} = \frac{\partial u_i}{\partial x_j}$$

What are these ε_{ij}, $i \neq j$?

Shear strains

$$\varepsilon_{12} = \frac{\partial u_1}{\partial x_2}$$



$$\varepsilon_{21} = \frac{\partial u_2}{\partial x_1}$$

$$\begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$

Strain tensor, (symmetric)

→ 6 Indep. components

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Total Differential of Force

$$dF_i = \frac{\partial F_i}{\partial A_j} dA_j = \sum_j \sigma_{ij}\, dA_j = \sigma_i^{\ j}\, dA_j$$

$$\begin{pmatrix} dF_1 \\ dF_2 \\ dF_3 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} dA_1 \\ dA_2 \\ dA_3 \end{pmatrix}$$

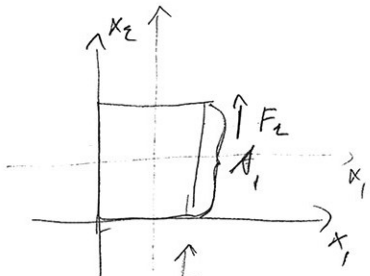
Stress tensor

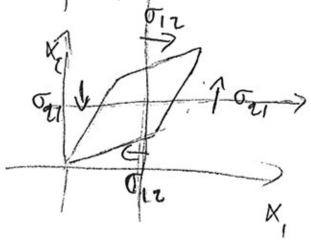
Symmetric → 6 Indep. comp.

Shear stresses

$\sigma_{ij}\ ,\ i \neq j$

$$\sigma_{ij} = \frac{\partial F_i}{\partial A_j}$$





ON-AXIS

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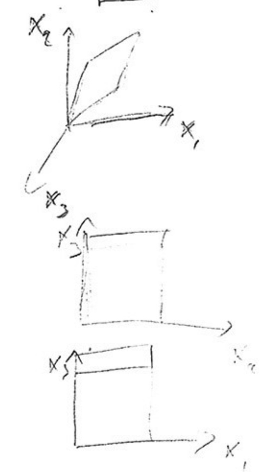
Stress-strain relations for orthogonal axis of symmetry = $\boxed{\text{ORTHOTROPY}}$

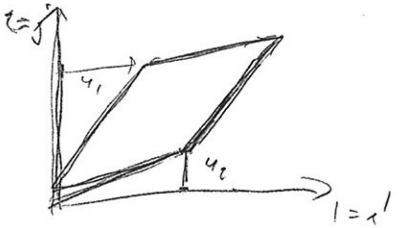
	σ_{11}	σ_{22}	σ_{33}	$2\sigma_{12}$	$2\sigma_{13}$	$2\sigma_{23}$
ε_{11}	S_{1111}	S_{1122}	S_{1133}			
ε_{22}	S_{2211}	S_{2222}	S_{2233}			
ε_{33}	S_{3311}	S_{3322}	S_{3333}			
ε_{12}				S_{1212}		
ε_{13}	Symmetric				S_{1313}	
ε_{23}						S_{2323}

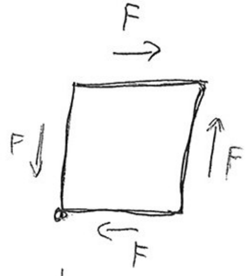


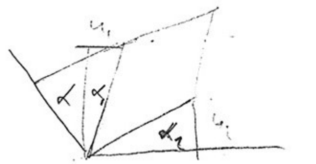
	ε_{11}	ε_{22}	ε_{33}	$2\varepsilon_{12}$	$2\varepsilon_{13}$	$2\varepsilon_{23}$
σ_{11}	Q_{1111}	Q_{1122}	Q_{1133}			
σ_{22}	Q_{2211}	Q_{2222}	Q_{2233}			
σ_{33}	Q_{3311}	Q_{3322}	Q_{3333}			
σ_{12}				Q_{1212}		
σ_{13}	Symmetric				Q_{1313}	
σ_{23}						Q_{2323}

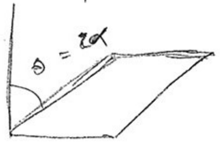
Measurement of shear strain

11.9.06 ⑦









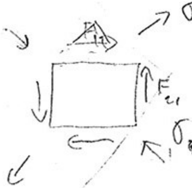
$$\varepsilon_{i'j'} = \varepsilon_{j'i'}$$

$$\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} = \tan\alpha_1 = \tan\left(\frac{\theta}{2}\right)$$

$$\varepsilon_{21} = \frac{\partial u_2}{\partial x_1} = \tan\alpha_2 = \tan\frac{\theta}{2}$$

$$\varepsilon_{12} + \varepsilon_{21} = 2\varepsilon_{12} = \tan\alpha_1 + \tan\alpha_2$$
$$= 2\tan\alpha$$
$$= 2\tan\left(\frac{\theta}{2}\right)$$

Measurement of shear stress



$$\sigma_T = \frac{F_T}{A_T} \Rightarrow F_T = \sigma_T A_T$$

$$\sigma_C = \frac{F_C}{A_C}$$

$$F_T = \cos 45^\circ F_{12} + \cos 45^\circ \cdot F_{21} = \frac{2}{\sqrt{2}} F_{12} = \sqrt{2} F_{12}$$

$$\sigma_{12} = \frac{F_{12}}{A_{12}} = \frac{F_{12}}{2 \cdot \cos 45^\circ \frac{A_T}{2}} = \frac{F_{12}}{\frac{1}{\sqrt{2}} A_T} = \frac{F_T}{\sqrt{2} A_T} = \sigma_T$$

$$\boxed{\sigma_{12} + \sigma_{21} = \sigma_T}$$

$$F_c = \sin 45^\circ F_{21} + \sin 45^\circ F_{12} = \sqrt{2} F_{12}$$

$$\sigma_{12} = \frac{F_{12}}{A_{12}} = \frac{F_{12}}{2 \cdot \cos 45^\circ \frac{A_c}{2}} = \frac{F_{12}}{\frac{1}{\sqrt{2}} A_c} = \frac{F_c}{A_c} = \sigma_c$$

Tulos oikea, sama saadaan jännityksen muunnoksella

$$\begin{pmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{pmatrix}$$

Shear compliance

$$S_{i'j'i'j'} = \frac{\varepsilon_{i'j'}}{2\sigma_{i'j'}} = \frac{\tan\alpha}{\sigma_T} = \frac{\tan\frac{\theta}{2}}{\sigma_c}$$

Shear stiffness

$$Q_{i'j'i'j'} = \frac{\sigma_{i'j'}}{2\varepsilon_{i'j'}} = \frac{1}{4}\frac{1}{S_{i'j'i'j'}}$$

Uniaxial stress experiments $\boxed{\text{in } j}$

4.9.06 ⑧

$$\varepsilon_{ii} = S_{iijj}\sigma_{jj} \quad (\text{no sum})$$

1° $i=j$: $\sigma_{jj} = E_j \varepsilon_{jj}$

$$\Rightarrow \boxed{S_{jjjj} = \frac{1}{E_j}}$$

2° $i \neq j$

$$\varepsilon_{ii} = -\nu_{ij}\varepsilon_{jj} = -\frac{\nu_{ij}}{E_j}\sigma_{jj} \Rightarrow \boxed{S_{iijj} = -\frac{\nu_{ij}}{E_j}}$$

Pure Shear Experiments

$$\varepsilon_{ij} = S_{ijij}\, 2\sigma_{ij}$$

$$\sigma_{ij} = G_{ij} 2\varepsilon_{ij} = Q_{ijij}\, 2\varepsilon_{ij}$$

$$\Rightarrow \boxed{S_{ijij} = \frac{1}{4G_{ij}}} = \frac{1}{4Q_{ijij}}$$

Engineering constants

$$E_j = \frac{1}{S_{jjjj}}$$

$$\nu_{ij} = -\frac{S_{iijj}}{S_{jjjj}} = -S_{iijj}E_j$$

$$G_{ij} = \frac{1}{4S_{ijij}}$$

Symmetry of compliance matrix!

$$S_{iijj} = S_{jjii}$$

$$\Rightarrow \boxed{\frac{\nu_{ij}}{E_j} = \frac{\nu_{ji}}{E_i}}$$

ON-Axis

ORTHOTROPIC RELATIONS

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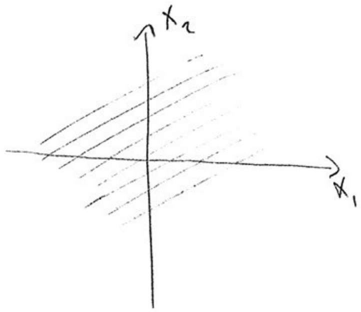
OFF-Axis mechanical behavior



(Determination of compliance:)
(OFF-axis stresses known
→ determine off-axis strains)

1° Stress transformation to on-axis

2° Use on-axis compliance matrix
→ on-axis strains

3° Negative transformation of strain to off-axis

Determine off-axis stress
(strains known)

1° Strain transformation to on-axis

2° Use stiffness matrix

3° Negative transformation of stress

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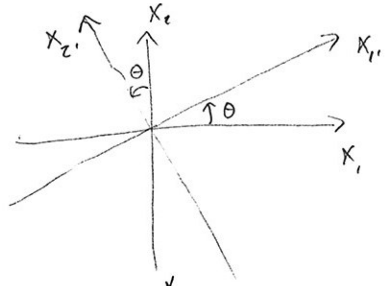
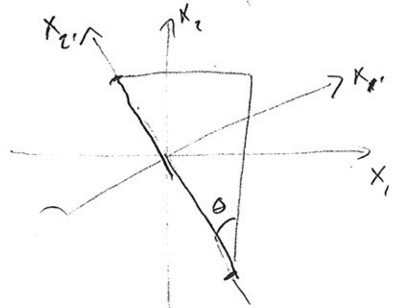
Co-ordinate transformation of stress



$$\boxed{dF_{i'} = \sum_j \frac{\partial F_{i'}}{\partial A_j} dA_j}$$

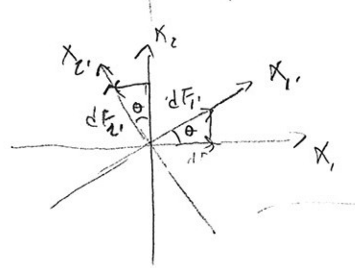
$$dF_1 = \frac{\partial F_1}{\partial A_1} dA_1 + \frac{\partial F_1}{\partial A_2} dA_2$$
$$= \sigma_{11}\, d(\cos\theta\, A_{1'}) + \sigma_{12}\, d(\sin\theta\, A_{1'})$$
$$= \sigma_{11} \cos\theta\, dA_{1'} + \sigma_{12} \sin\theta\, dA_{1'}$$



$$dF_1 = \cos\theta\, dF_{1'} - \sin\theta\, dF_{2'}$$
$$= \cos\theta\left[\sigma_{1'1'} dA_{1'} + \sigma_{1'2'} dA_{2'}\right] - \sin\theta\left[\sigma_{2'1'} dA_{1'} + \sigma_{2'2'} dA_{2'}\right]$$

$$\frac{dF_1}{dA_{1'}} = \cos\theta\, \sigma_{11} + \sin\theta\, \sigma_{12}$$

$$\frac{dF_1}{dA_{1'}} = \cos\theta\, \sigma_{1'1'} - \sin\theta\, \sigma_{2'1'}$$



$$dF_2 = \sigma_{21} dA_1 + \sigma_{22} dA_2 = \cos\theta\, \sigma_{21} dA_{1'} + \sin\theta\, \sigma_{22} dA$$

$$dF_2 = \sin\theta\, dF_{1'} + \cos\theta\, dF_{2'}$$
$$= \sin\theta\left[\sigma_{1'1'} dA_{1'} + \sigma_{1'2'} dA_{2'}\right] + \cos\theta\left[\sigma_{2'2'} dA_{2'} + \sigma_{2'1'} dA_{1'}\right]$$

$$\boxed{\begin{aligned} \frac{dF_2}{dA_{1'}} &= \cos\theta\, \sigma_{21} + \sin\theta\, \sigma_{22} \\ \frac{dF_2}{dA_{1'}} &= \sin\theta\, \sigma_{1'1'} + \cos\theta\, \sigma_{2'1'} \end{aligned}}$$

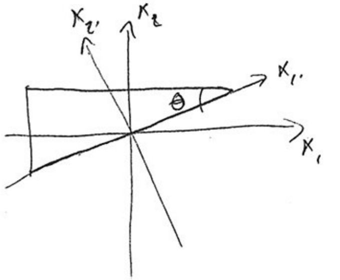
$$\boxed{\begin{aligned} m\sigma_{11} + n\sigma_{12} &= m\sigma_{1'1'} - n\sigma_{1'2'} \\ n\sigma_{22} + m\sigma_{12} &= n\sigma_{1'1'} + m\sigma_{1'2'} \end{aligned}}$$

Solve

$\Rightarrow \sigma_{1'1'} = \ldots$

$\sigma_{1'2'} = \ldots$

Crd transform of stress cntd 5.9.06 (11)



$$dF_1 = \sigma_{11}\, dA_1 + \sigma_{12}\, dA_2 = \sigma_{11} \sin\theta\, dA_{2'} + \sigma_{12} \cos\theta\, dA_{2'}$$

$$\frac{dF_1}{dA_{2'}} = \sigma_{11} \sin\theta + \sigma_{12} \cos\theta$$

$$\frac{dF_1}{dA_{2'}} = \cos\theta\, \sigma_{1'2'} - \sin\theta\, \sigma_{2'2'}$$

$$n\sigma_{11} + m\sigma_{12} = m\sigma_{1'2'} - n\sigma_{2'2'}$$

$$\Rightarrow \sigma_{2'2'} = \frac{m}{n}(\sigma_{1'2'} - \sigma_{12}) - \sigma_{11} \Rightarrow \left.\begin{matrix}\sigma_{1'1'}\\ \sigma_{2'2'}\\ \sigma_{1'2'}\end{matrix}\right\} \text{All known:}$$

	σ_{11}	σ_{22}	σ_{12}
$\sigma_{1'1'}$	m^2	n^2	$2mn$
$\sigma_{2'2'}$	n^2	m^2	$-2mn$
$\sigma_{1'2'}$	$-mn$	mn	m^2-n^2

Invariant: $\sigma_{1'1'} + \sigma_{2'2'} = (m^2+n^2)(\sigma_{11}+\sigma_{22}) = \sigma_{11} + \sigma_{22}$

Check Inversibility of stress transformation 6.9.06 (12)

	$m^2\sigma_{11} + n^2\sigma_{22} - 2mn\sigma_{12}$	$n^2\sigma_{11} + m^2\sigma_{22} + 2mn\sigma_{12}$	$mn\sigma_{11} - mn\sigma_{22} + (m^2-n^2)\sigma_{12}$
σ_{11}	m^2	n^2	$2mn$
σ_{22}	n^2	m^2	$-2mn$
σ_{12}	$-mn$	mn	m^2-n^2

$$\sigma_{11} = m^4\sigma_{11} + m^2n^2\sigma_{22} - 2m^3n\sigma_{12} + n^4\sigma_{11} + m^2n^2\sigma_{22} + 2mn^3\sigma_{12} + 2m^2n^2\sigma_{11} - 2m^2n^2\sigma_{22} + 2(m^3n - mn^3)\sigma_{12} = (m^2+n^2)^2\sigma_{11} = \sigma_{11}$$

$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{pmatrix} = (R)(R^{-1}) \begin{matrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{matrix}$$

$$\begin{pmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{pmatrix} \begin{pmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2-n^2 \end{pmatrix} = \begin{pmatrix} m^4+n^4+2m^2n^2 & m^2n^2+m^2n^2-2m^2n^2 & -2m^3n+2mn^3+2m^3n-2mn^3 \\ n^2m^2+m^2n^2-2m^2n^2 & n^4+m^4+2m^2n^2 & -2mn^3+2m^3n-2m^3n+2mn^3 \\ -m^3n+mn^3+m^3n-mn^3 & -mn^3+m^3n-m^3n+mn^3 & 2m^2n^2+2m^2n^2+m^4-m^2n^2-m^2n^2+n^4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Co-ordinate transformations of strain

6.9.06

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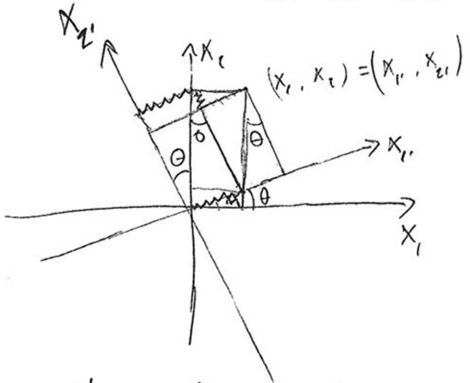
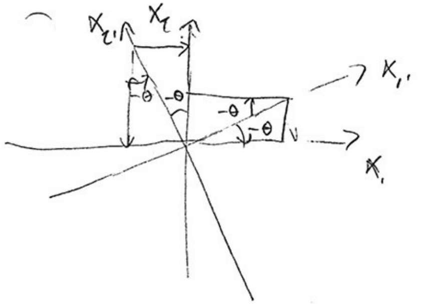


$$x_1' = \cos\theta\, x_1 + \sin\theta\, x_2$$

$$x_2' = \cos\theta\, x_2 - \sin\theta\, x_1$$



$$x_1 = \cos\theta\, x_{1'} + \sin(-\theta)\, x_{2'}$$
$$= \cos\theta\, x_{1'} - \sin\theta\, x_{2'}$$

$$x_2 = \cos\theta\, x_{2'} + \sin\theta\, x_{1'}$$
$$= -\sin\theta\, x_{1'} + \cos\theta\, x_{2'}$$

$$u_{1'} = m u_1 + n u_2$$
$$u_{2'} = -n u_1 + m u_2$$
$$u_1 = m u_{1'} - n u_{2'}$$
$$u_2 = n u_{1'} + m u_{2'}$$

$$\varepsilon_{i'j'} = \frac{\partial u_{i'}}{\partial x_{j'}} \qquad \varepsilon_{i'j'} = \frac{\partial u_{i'}}{\partial x_{j'}}$$

$$\varepsilon_{i'j'} = \frac{\partial u_{i'}}{\partial x_j}\frac{\partial x_j}{\partial x_{j'}} \qquad \text{sum!}$$

$$\varepsilon_{1'1'} = \frac{\partial u_{1'}}{\partial x_1}\frac{\partial x_1}{\partial x_{1'}} + \frac{\partial u_{1'}}{\partial x_2}\frac{\partial x_2}{\partial x_{1'}}$$
$$= \frac{\partial u_{1'}}{\partial x_1} m + \frac{\partial u_{1'}}{\partial x_2} n$$
$$= \left[m\frac{\partial u_1}{\partial x_1} + n\frac{\partial u_2}{\partial x_1}\right] m + \left[m\frac{\partial u_1}{\partial x_2} + n\frac{\partial u_2}{\partial x_2}\right] n$$
$$= \underline{m^2\varepsilon_{11} + 2nm\,\varepsilon_{12} + n^2\varepsilon_{22}}$$

$$\varepsilon_{2'2'} = \frac{\partial u_{2'}}{\partial x_{2'}} = \frac{\partial u_{2'}}{\partial x_1}\frac{\partial x_1}{\partial x_{2'}} + \frac{\partial u_{2'}}{\partial x_2}\frac{\partial x_2}{\partial x_{2'}}$$
$$= \frac{\partial u_{2'}}{\partial x_1}(-n) + \frac{\partial u_{2'}}{\partial x_2} m$$
$$= \left[-n\frac{\partial u_1}{\partial x_1} + m\frac{\partial u_2}{\partial x_1}\right](-n) + \left[-n\frac{\partial u_1}{\partial x_2} + m\frac{\partial u_2}{\partial x_2}\right] m$$
$$= \underline{n^2\varepsilon_{11} - 2nm\,\varepsilon_{12} + m^2\varepsilon_{22}}$$

$$\varepsilon_{1'2'} = \frac{\partial u_{1'}}{\partial x_{2'}} = \frac{\partial u_{1'}}{\partial x_1}\frac{\partial x_1}{\partial x_{2'}} + \frac{\partial u_{1'}}{\partial x_2}\frac{\partial x_2}{\partial x_{2'}}$$
$$= \frac{\partial u_{1'}}{\partial x_1}(-n) + \frac{\partial u_{1'}}{\partial x_2} m$$
$$= \left[m\frac{\partial u_1}{\partial x_1} + n\frac{\partial u_2}{\partial x_1}\right](-n) + \left[m\frac{\partial u_1}{\partial x_2} + n\frac{\partial u_2}{\partial x_2}\right] m$$
$$= -mn\,\varepsilon_{11} + (m^2 - n^2)\,\varepsilon_{12} + mn\,\varepsilon_{22}$$

Checking the strain transform matrix

6.9.06

(14)

$$\begin{pmatrix} m^2 & n^2 & 2nm \\ n^2 & m^2 & -2nm \\ -mn & mn & m^2-n^2 \end{pmatrix}\begin{pmatrix} m^2 & n^2 & -2nm \\ n^2 & m^2 & 2nm \\ mn & -mn & m^2-n^2 \end{pmatrix} =$$

$$\begin{pmatrix} m^4+n^4+2m^2n^2 & m^2n^2+m^2n^2-2m^2n^2 & -2m^3n+2n^3m+2m^3n-2n^3m \\ n^2m^2+n^2m^2-2n^2m^2 & n^4+m^4+2m^2n^2 & -2n^3m+2m^3n-2m^3n+2n^3m \\ -m^3n-n^3m+m^3n-n^3m & -n^3m+m^3n-m^3n+n^3m & 2m^2n^2+2m^2n^2+m^4+n^4-2m^2n^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \%$$

6.9.06 (15)

Co-ordinate transformations of compliance and stiffness

ON-AXIS → OFF-AXIS

	S_{1111}	S_{2222}	S_{3333}	S_{1122}	S_{1133}	S_{2233}	S_{1212}	S_{1313}	S_{2323}
S'_{1111}	□	□		□			□		
S'_{2222}	□	□		□			□		
S'_{3333}									
S'_{1122}	□	□		□			□		
S'_{1133}									
S'_{2233}									
S'_{1212}	□	□		□			□		
S'_{1313}									
S'_{2323}									
S'_{1112}	□	□		□			□		
S'_{1113}									
S'_{2212}	□	□		□			□		
S'_{2223}									
S'_{3313}									
S'_{3323}									
S'_{1213}									
S'_{1223}									
S'_{1323}									

1° Transform stress to on-axis

2° Use on-axis compliances to compute on-axis strains

3° Transform strain to off-axis

4° Identify off-axis compliance matrix elements

5° Identify coefficients of any element wrt. any on-axis element

7.9.06 (16)

Time-dependent behavior

$$\varepsilon_{ij} = \text{function}(\sigma_{kl})$$

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} = \sum_k \sum_l S_{ijkl}\sigma_{kl}$$

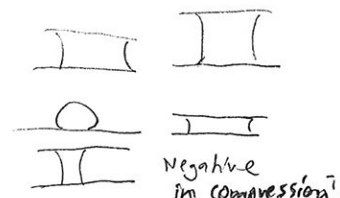
$$= S_{ij11}\sigma_{11} + S_{ij22}\sigma_{22} + S_{ij33}\sigma_{33} + \underbrace{2S_{ij12}\sigma_{12}}_{S_{ij12}\sigma_{12} + S_{ij21}\sigma_{21}} + 2S_{ij13}\sigma_{13} + 2S_{ij23}\sigma_{23}$$

$$\sigma_{ij} = \text{function}(\varepsilon_{kl})$$

$$\sigma_{ij} = Q_{ijkl}\varepsilon_{kl}$$

Is this right?

What is the Youngs modulus $\frac{1}{S_{iiii}}$ of water?



Negative in compression!

$$\sigma_{ij} = \text{function}(\dot{\varepsilon}_{kl}) = \text{function}\left(\frac{d\varepsilon_{kl}}{dt}\right)$$

Newtonian viscosity: $\sigma_{ij} = \eta_{ijkl}\dot{\varepsilon}_{kl}\delta_{ik}\delta_{jl}, \quad i \neq j$

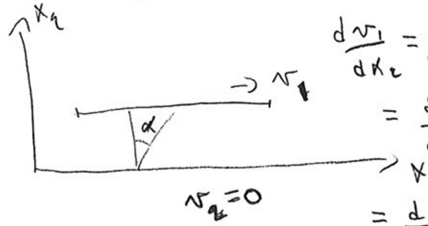
$$\sigma_{ij} = \frac{\partial F_i}{\partial x_j} \qquad \varepsilon_{kl} = \frac{\partial u_k}{\partial x_l}$$

Isotropic fluid: $\sigma_{ij} = \eta \frac{dv_i}{dx_j}, \quad i \neq j \quad \rightarrow = \mu \frac{d\tan\alpha}{dt}$

$$= \frac{1}{\cos^2\alpha}\frac{d\alpha}{dt} \approx \frac{d\alpha}{dt}$$

Generally: $\sigma_{ij} = Q_{ijkl}\varepsilon_{kl} + \eta_{ijkl}\dot{\varepsilon}_{kl}$?

How about ε_{ij} ?



$$\frac{dv_1}{dx_2} = \frac{d}{dx_2}\frac{d\tan\alpha \cdot x_2}{dt} = \frac{dx_2}{dx_2}\cdot\frac{d\tan\alpha}{dt} = \frac{d\tan\alpha}{dt}$$

$$= \frac{d\sin\alpha\cos^{-1}\alpha}{d\alpha}\frac{d\alpha}{dt} = \left(\frac{\cos\alpha}{\cos\alpha} + \frac{\sin^2\alpha}{\cos^2\alpha}\right)\frac{d\alpha}{dt}$$

$$= \frac{1}{\cos^2\alpha}\frac{d\alpha}{dt}$$

12.9.06 (17)

The Maxwell model

$$\varepsilon = \varepsilon^{e} + \varepsilon^{i}$$

$$\sigma = Q\varepsilon^{e}$$

$$\sigma = \eta\,\dot{\varepsilon}^{i}$$

$$\dot{\varepsilon}^{i} = \frac{d\varepsilon^{i}}{dt} = \frac{Q\varepsilon^{e}}{\eta} = \frac{\sigma}{\eta}$$

Steady creep, if σ constant

$$\varepsilon^{i} = \frac{\sigma}{\eta} t$$

$$\frac{d\varepsilon}{dt} = \frac{d\varepsilon^{e}}{dt} + \frac{d\varepsilon^{i}}{dt}$$

$$= \frac{d\sigma}{Q\,dt} + \frac{\sigma}{\eta}$$

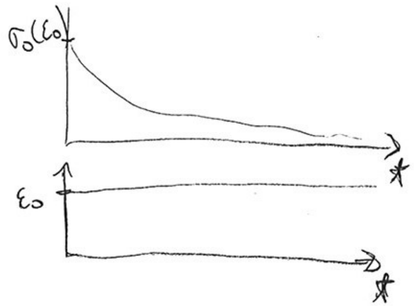
Stress relaxation experiment:

$$\frac{d\varepsilon}{dt} = 0 = \frac{d\sigma}{Q\,dt} + \frac{\sigma}{\eta}$$

$$\frac{d\sigma}{\sigma} = -\frac{Q}{\eta} dt \quad \Big| \int$$

$$\ln\sigma = -\frac{Q}{\eta} t + C$$

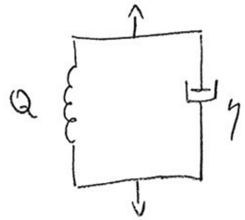
$$\sigma = \sigma_0 e^{-\frac{Q}{\eta} t} = \sigma_0 e^{-\frac{t}{\tau}}$$



$\tau \equiv$ relaxation time $\equiv \frac{\eta}{Q}$

12.9.06 (18)

The Kelvin model



$$\sigma = Q\varepsilon + \eta\dot{\varepsilon}$$

$$\frac{\varepsilon}{\sigma} = \frac{1}{Q} - \frac{\eta}{\sigma Q}\frac{d\varepsilon}{dt}$$

$$\frac{\varepsilon}{\sigma} - \frac{1}{Q} = -\frac{\tau}{\sigma}\frac{d\varepsilon}{dt}$$

$$\frac{1}{\frac{\varepsilon}{\sigma} - \frac{1}{Q}} = -\frac{\sigma}{\tau}\frac{dt}{d\varepsilon}$$

$$\frac{d\varepsilon}{\varepsilon - \frac{\sigma}{Q}} = -\frac{dt}{\tau}$$

Creep experiment:

σ = constant

$$\frac{d\ln\left(\varepsilon - \frac{\sigma}{Q}\right)}{d\varepsilon} = \frac{d\ln(\ \ )}{d\left(\varepsilon - \frac{\sigma}{Q}\right)}\frac{d\left(\varepsilon - \frac{\sigma}{Q}\right)}{d\varepsilon}$$

$$= \frac{1}{\varepsilon - \frac{\sigma}{Q}}$$

$$\frac{d\varepsilon}{\varepsilon - \frac{\sigma}{Q}} = -\frac{dt}{\tau} \quad \Big| \int$$

$$\ln\left(\frac{\varepsilon}{\sigma} - \frac{1}{Q}\right) = -\frac{t}{\tau} + C$$

$$\varepsilon - \frac{\sigma}{Q} = c_2 e^{-\frac{t}{\tau}}$$

$$\varepsilon = \frac{\sigma}{Q} + c_2 e^{-\frac{t}{\tau}}$$

$$= \varepsilon_\infty + (\varepsilon_0 - \varepsilon_\infty) e^{-\frac{t}{\tau}}$$

$$= \underbrace{\varepsilon_0}_{=0} e^{-\frac{t}{\tau}} + \varepsilon_\infty\left(1 - e^{-\frac{t}{\tau}}\right)$$

$$= \varepsilon_\infty\left(1 - e^{-\frac{t}{\tau}}\right)$$

$$\varepsilon = \sigma S_\infty\left(1 - e^{-\frac{t}{\tau}}\right)$$

$\tau \equiv$ Retardation time

13.9.06

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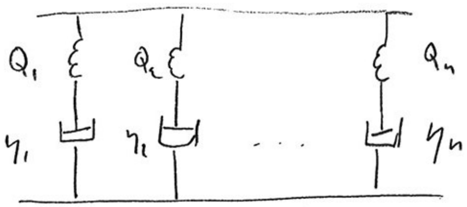
Generalized Maxwell model



Stress relaxation experiment

$$\sigma = \sum_i \sigma_i = \sum_{i=1}^{n} \sigma_{0i} e^{-\frac{t}{\tau_i}} = \sum_{i=1}^{n} Q_i \varepsilon e^{-\frac{t}{\tau_i}} = \varepsilon \sum_{i=1}^{n} Q_i e^{-\frac{t}{\tau_i}}$$

$\sigma(t) = \varepsilon R(t)$ $\quad$ τ_i:s with weights Q_i:

Relaxation $\begin{cases}\text{function} \\ \text{modulus}\end{cases}$

If one of the Maxwell elements is degenerate, $\frac{\eta}{Q} = \infty$

$$\sigma = \varepsilon \left[Q_\infty + \sum_{i=1}^{n} Q_i e^{-\frac{t}{\tau_i}} \right] = \sigma_\infty + \varepsilon \sum_{i=1}^{n} Q_i e^{-\frac{t}{\tau_i}}$$

Now, this was for instantly applied (at time 0) strain, which was then held constant. Similarly, differential strain at time t' induces differential stress at time t:

$$d\sigma(t) = d\varepsilon(t')\, R(t-t')$$

Total stress with arbitrary strain history

$$\sigma(t) = \int_{-\infty}^{t} \frac{d\sigma}{dt'} dt' = \int_{-\infty}^{t} R(t-t') \frac{d\varepsilon(t')}{dt'} dt' = \int_{t'=-\infty}^{t'=t} R(t-t')\, d\varepsilon(t')$$

(Linear viscoelasticity: $\begin{cases}\text{Stresses, strains additive; } \exists \text{ unique Relaxation function} \\ \text{Relaxation function can be determined} \\ \text{in a single stress relaxation experiment}\end{cases}$)

$$R(t) = Q_\infty + \sum_{i=1}^{n} Q_i e^{-\frac{t}{\tau_i}} = Q_\infty + \int_0^\infty Q(\tau) e^{-\frac{t}{\tau}} d\tau = Q_\infty + \int_{-\infty}^{\infty} Q(\tau)\tau e^{-\frac{t}{\tau}} d(\ln\tau)$$

$$= \boxed{Q_\infty} + \int_{-\infty}^{\infty} \boxed{H(\tau)} e^{-\frac{t}{\tau}} d(\ln\tau)$$

$d\ln\tau = \frac{d\tau}{\tau}$

Equilibrium modulus, Rubbery ~Steady

Relaxation spectrum

Glassy Modulus
$$R(t=0) = Q_\infty + \int_{-\infty}^{\infty} H(\tau)\, d(\ln\tau) \equiv R_e$$

18.9.06

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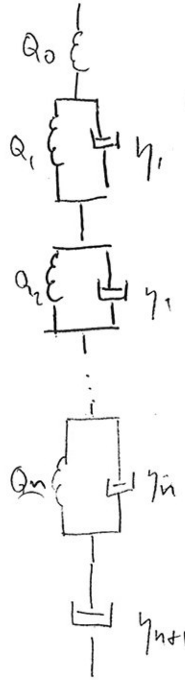
Generalized Voigt-Kelvin Model



Creep experiment

$$\varepsilon(t) = \sum_i \varepsilon_i = \sum_i \sigma S_{\infty i} \left(1 - e^{-\frac{t}{\tau_i}}\right)$$

$$= \sigma \left[\frac{1}{Q_0} + \sum_{i=1}^{n} S_{\infty i} \left(1 - e^{-t/\tau_i}\right) + \frac{t}{\eta_{n+1}} \right]$$

$$= \sigma \left[S_0 + \sum_{i=1}^{n} S_{\infty i} \left(1 - e^{-t/\tau_i}\right) + \frac{t}{\eta} \right]$$

$$\varepsilon(t) = \sigma J(t)$$

Creep $\begin{cases}\text{function} \\ \text{compliance}\end{cases}$

$$d\varepsilon(t) = d\sigma(t')\, J(t-t')$$

$$\varepsilon(t) = \int_{-\infty}^{t} \frac{d\varepsilon}{dt'} dt' = \int_{-\infty}^{t} J(t-t') \frac{d\sigma(t')}{dt'} dt'$$

$$= \int_{t'=-\infty}^{t'=t} J(t-t')\, d\sigma(t')$$

$$J(t) = S_0 + \int_0^\infty S(\tau) \left(1 - e^{-t/\tau}\right) d\tau + \frac{t}{\eta}$$

$$= S_0 + \int_{-\infty}^{\infty} S(\tau) \left(1 - e^{-t/\tau}\right) \tau\, d(\ln\tau) + \frac{t}{\eta}$$

$$= \boxed{S_0} + \int_{-\infty}^{\infty} \boxed{L(\tau)} \left(1 - e^{-t/\tau}\right) d(\ln\tau) + \frac{t}{\boxed{\eta}}$$

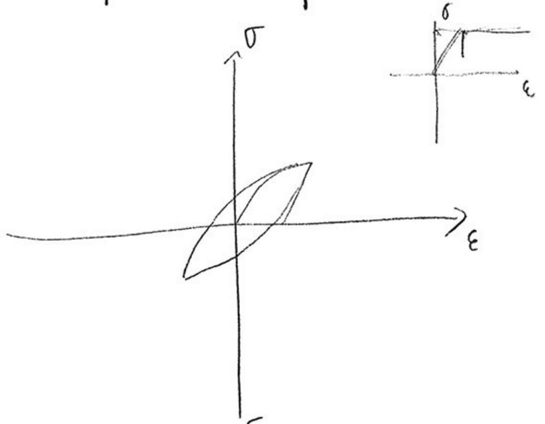
Glassy compliance $\quad$ Retardation spectrum $\quad$ Steady-state viscosity

$\begin{cases}\text{Steady-state} \\ \text{Equilibrium} \\ \text{Rubbery}\end{cases}$ compliance? $\quad$ Only if $\eta \to \infty$

$$J(\infty) = S_0 + \int_{-\infty}^{\infty} L(\tau)\, d(\ln\tau)$$

Dynamic experiments

14.9.06 (21)

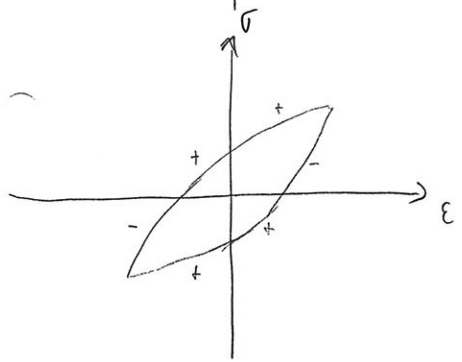


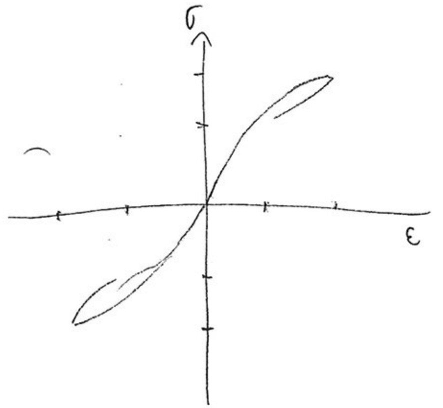
$$W = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}Q\varepsilon^2 = \quad \sigma^2$$

$$\frac{dW}{d\varepsilon} = Q\varepsilon = \sigma$$

$$dW = F\,du$$

$$dW = \sigma\, d\varepsilon$$





14.9.06 (22)

Stress-strain-time-temperature-relations
- moisture

Crosslinked polymers: Equilibrium elasticity exists

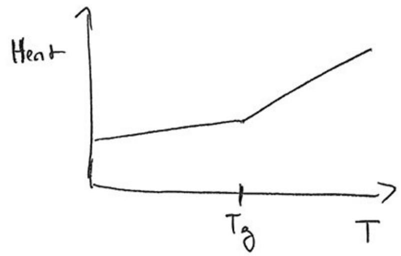
$$\sigma = \varepsilon\left[Q_\infty + \sum_{i=1}^{n} Q_i e^{-\frac{t}{\tau_i}}\right] \Rightarrow R(t-t') = Q_\infty \left(+ \sum_{i=1}^{n} Q_i e^{-\frac{t-t'}{\tau_i}} \right)$$

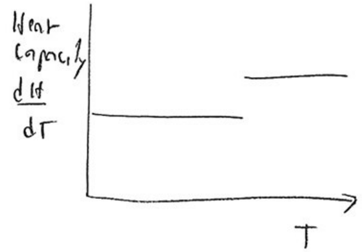
noncrosslinked: liquid-like flow $(Q_\infty = 0)$

$\eta_{nn} < \infty$

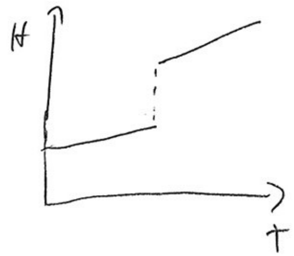
Amorphous polymers:

Glass transition   second-order transition





Crystalline polymers: melting   First-order transition



1° Large-deformation equilibrium properties

2° Small-deformation non eq (viscoelastic) –"–

(3° Large-deformation time-dependent)

14.9.06

(23)

1° Large-deformation Eq.
From kinetic theory:

$$\sigma = \rho k T (\varepsilon - \varepsilon^{-2})$$

(Engineering
Tensile stress for engineering tensile strain)

Shear modulus $G = \rho k T$

How to (approach this behavior?
(reach the equilibrium:

- increase time
  $\cong$ reduce straining rate
- Speed up relaxation (Temperature, ...

2° Small-deformation non-eq.
- see above

18.9.06

(24)

Time - temperature equivalency
Moisture Content - Specific volume

1° All characteristic times similarly affected by temperature change.
→ temperature function $a_T = \tau/\tau_0$

2° $Q, \eta \propto T, \rho$

⇒ Data at all Temperatures, Densities, superposes

$$\log \frac{R(t)\, T_0 \rho_0}{T \rho}$$

$$\log t/a_T = \log\left(t \frac{\tau_0}{\tau}\right)$$

BUT what is a_T?

1° Fit the curves to superpose

1a 2° $a_T \approx \frac{\eta T_0 \rho_0}{\eta_0 T \rho} \approx \frac{\tau}{\tau_0}$ since $\frac{\eta}{\eta_0} \approx \frac{\tau T \rho}{\tau_0 T_0 \rho_0}$ $\quad \eta \propto \tau, T, \rho$

WLF: $\log a_T \approx \frac{-8{,}86\,(T - T_s)}{101{,}6 + T - T_s}$ $\quad T_s \approx T_g + 50$

$$= -\frac{C_1 (T - T_g)}{C_2 + T - T_g}$$

Moisture content?
Density / specific volume?

Elastoplasticity

20.9.06

25

$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p$

$\sigma_{ij} = Q_{ijkl}\,\varepsilon^e_{kl} = Q_{ijkl}\,(\varepsilon_{kl} - \varepsilon^p_{kl})$

Incompressible plastic strain: $\varepsilon^p_{ii} = 0$

Yied function $f = f(\sigma_{ij})$

Yield condition $f = 0$ → Yield surface

Isotropic hardening
- yield surface expands

Kinematic hardening
- yield surface translates

No stress state can exist outside of yield surface: $f \le 0$

$f < 0$: Elastic range

$f = 0$: Plastic deformation may occur

Yield surface gradient: $\nabla f = \bar{e}_i \frac{\partial}{\partial x_i} f$

in stress space: $\nabla f = \bar{e}_{ij} \frac{\partial}{\partial \sigma_{ij}} f$

Gradient component $\frac{\partial f}{\partial \sigma_{ij}}$

Plastic loading: $f = 0$, $\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} > 0$

$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} < 0$ ⇒ Elastic unloading

$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = 0$ ⇒ Neutral loading

Elastoplasticity 2

20.9.06

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Example of yield criterion:

von Mises $f = \frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right] - k^2 = 0$

Uniaxial principal stress: $\frac{2}{6}\sigma_y^2 - k^2 = 0$ ⇒ $k = \pm\frac{\sigma_y}{\sqrt{3}}$

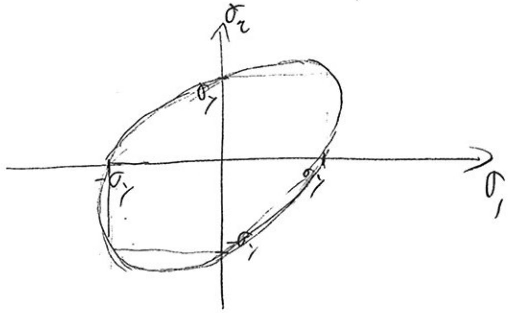
Pure shear $\sigma_3 = 0$, $\sigma_1 = -\sigma_2 = \sigma$

$\frac{1}{6}\left[(2\sigma)^2 + 2\sigma^2\right] = k^2$ ⇒ $\pm\sigma_y = k$

Equi-Biaxial principal stress: $\frac{1}{6} 2\sigma^2 = k^2$ ⇒ $\pm\frac{\sigma_y}{\sqrt{3}} = k$

$\frac{1}{2}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right] = \sigma_y^2$

For plane stress, $\sigma_3 = 0$: $\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_y^2$



Isotropic Hardening:

σ_y = function(ε^p)

k = function(ε^p)

$f(\sigma_{ij}, k) \to f(\sigma_{ij}, k') = f(\sigma_{ij}, k + \delta k)$

δk = function(ε^p)

Kinematic hardening:

$f(\sigma_{ij}, k) \to f(\sigma'_{ij}, k) = f(\sigma_{ij} + \delta\sigma_{ij}, k)$

$\delta\sigma'_{ij}$ = function(ε^p)

Failure criterion

Failure criterion in stress space:

$$f(\sigma_{ij}) = 0$$

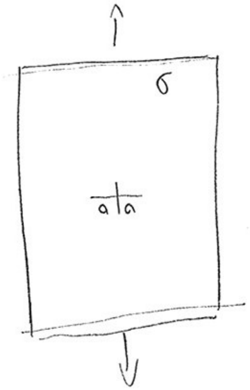
Failure criterion in strain space

$$g(\varepsilon_{ij}) = 0$$

Related by constitutive Eq.

Fracture Mechanics

21.9.06



$$\frac{d\Pi}{da} + \frac{dW_s}{da} \leq 0$$

$$\frac{dW_s}{da} \leq -\frac{d\Pi}{da} = \left|\frac{d\Pi}{da}\right|$$

$$\frac{dW_s}{da} = \frac{d\,4atG}{da} = 4tG$$

$$\frac{dW_s}{dA} = G$$

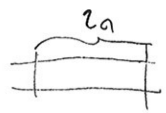
How to determine G?

Inglis

$$\Pi = \Pi_0 - \frac{\pi\sigma^2 a^2 t}{Q} \qquad W_s = 4atG$$

$$-\frac{d\Pi}{da} = \frac{2\pi\sigma^2 a t}{Q} \qquad \frac{dW_s}{da} = 4tG \qquad \frac{\pi\sigma_c^2 a}{2Q} = G \qquad \boxed{\text{LEFM}}$$

$$\sigma_c = \sqrt{\frac{2QG}{\pi a}}$$



$$W_e = \frac{1}{2} Q F_c$$

$$W_a = \int_{\delta_c}^{\delta_f} F\,d\delta \qquad \Delta W_s = W_e + W_a$$

$$G = \frac{W_e + W_a}{4at}$$

How about non-linear materials w. plastic yielding?

$$\sigma_c \approx \sigma_{pe} \neq \sqrt{\frac{2QG}{\pi a}}$$

Scaling:

$$\sigma_c = \frac{\sigma_{pe}}{\sqrt{1 + \frac{\pi a \sigma_{pe}^2}{2QG}}} = \frac{\sigma_{pe}}{\sqrt{1+\beta}}$$