On the Prediction of the Strength of Paper Structures with a Flaw

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Considering flaw sizes reported in the literature, linear elastic fracture mechanics does not seem to apply to the failure analysis of paper structures with a flaw. The failure stress of such structures cannot be predicted using fracture energy measurements or crack tip plasticity corrections. In contrast, a cohesive softening model gives good agreement with the measured failure stresses without adjustable parameters.

Compte tenu de la dimension des défauts rapportés dans la documentation, la mécanique de la rupture élastique linéaire ne semble pas s'appliquer à l'analyse des défaillances des structures de papier comportant des défauts. La contrainte de rupture de ces structures ne peut être prévue à l'aide des mesures de l'énergie de rupture ou des corrections de la plasticité de la zone à la tête de rupture. Par contre, un modèle de ramollissement cohésif est particulièrement en accord avec les contraintes de ruptures mesurées sans paramètres réglables.

INTRODUCTION

Since the establishment of quantitative fracture mechanics by Griffith in 1920 [1], various applications for paper have been introduced. According to the treatment by Irwin [2], the load-carrying capability of a linearly elastic structure with a flaw depends on one single material constant: *the critical stress intensity factor.* This approach has been applied to paper by Swinehart and Broek [3,4].

The assumptions behind the critical stress intensity factor approach are quite restrictive. One way to overcome these limitations is to define an "effective" or "equivalent" flaw size, which includes a plasticity correction to the actual physical flaw size. This approach, originally introduced by Irwin et al. [5], has been applied to paper by Andersson and Falk [6] and Donner [7].

The stress intensity approach is valid only for structures where the volume involving plastic deformation as well as the volume where the actual fracture process takes place are small in comparison to all the relevant dimensions of the structure. Where this is not true, the stress intensity factor characterization becomes inadequate. However, the *fracture energy,* defined as the work per unit crack area needed to create new traction-free surfaces, can be measured even in such cases. Quite a few paper applications have been presented [8–13].

Proper use of the fracture energy in the analysis of nonlinear structures is not straightforward. Often it requires a cohesive softening model – one has to know certain details of the material behaviour beyond the concept of fracture energy [14–16]. Tryding [17] has recently pioneered the application of cohesive softening models to paper.

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According to observations, the flaws that induce failure in paper webs often have the dimensions of 5–10 mm [4,18–20]. In this paper, we study the strength of paper specimens with such flaws. The critical nominal stress is measured and compared with the four fracture mechanics approaches mentioned above. We begin with the classical critical stress intensity factor. Then, we discuss the applicability of the linear elastic fracture mechanics (LEFM) approaches using the fracture energy and the plasticity correction. Finally, we introduce a cohesive softening model and use it to show why the other three approaches do not work.

EXPERIMENTAL OBSERVATIONS

Double-edge-notched specimens of a commercial 70 g/m^2 copy paper were strained to failure at rate 10%/min, temperature 23°C and relative humidity 50%, both in machine direction (MD) and cross-machine direction (CD). Three lengths of the edge notches, 2, 4 and 6 mm (total crack length 4, 8 and 12 mm) and two specimen widths (50 and 100 mm, the latter only in CD) were used. Specimen height was 1.5 times specimen width. Fifteen specimens of each geometry were tested.

The mean value of the critical nominal stress for the different specimens is given in Table I; the coefficient of variation was 3–10%. The critical nominal stress is the breaking load divided by the cross-sectional area of the specimen, prior to edge notching. Some mechanical properties of the paper material are given in Table II.

CRITICAL STRESS INTENSITY APPROACH

First we want to see if the experimental results are consistent with LEFM. For this purpose, we calculated the critical stress intensity factor from Eq. (1) [21],

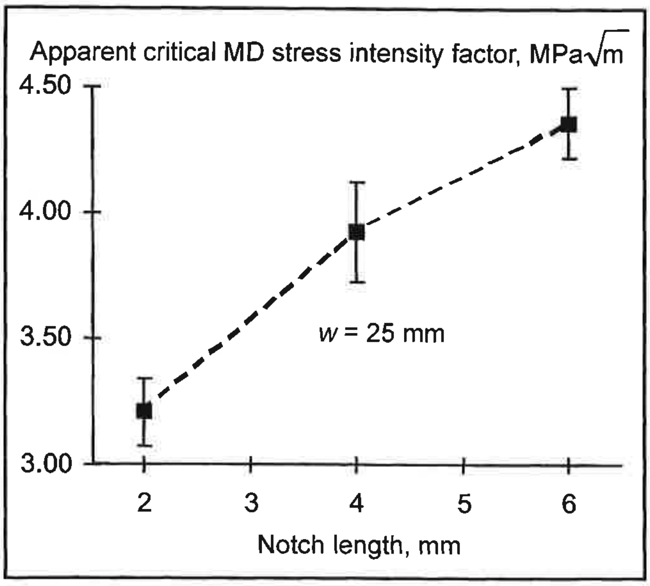

Fig. 1. Apparent critical stress intensity factor K_c (Eq. 1) in MD against notch length. The error bars show the 95% confidence limits.

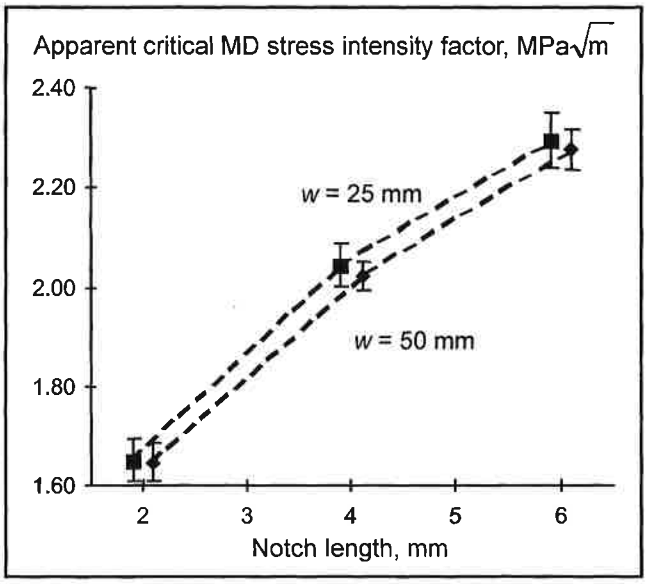

Fig. 2. Apparent critical stress intensity factor K_c (Eq. 1) in CD against notch length. Data for the two specimen widths are slightly displaced for clarity; they cannot be distinguished experimentally. The error bars show the 95% confidence limits.

TABLE I
CRITICAL NOMINAL STRESS (MPa) OF THE DIFFERENT SPECIMEN SIZES

	MD	CD	CD
	Half of specimen width, mm		
Notch length, mm	25	25	50
2	36.1	18.6	18.5
4	31.2	16.2	16.1
6	28.2	14.8	14.8

Equation 1

$$K_c = \sigma_c^N \beta \sqrt{\pi a}$$

$$\beta = \frac{1.122 - 0.561\frac{a}{w} - 0.205\left(\frac{a}{w}\right)^2 + 0.471\left(\frac{a}{w}\right)^3 + 0.190\left(\frac{a}{w}\right)^4}{\sqrt{1-\frac{a}{w}}}$$

TABLE II
MATERIAL PROPERTIES OF THE PAPER

	MD	CD
Young's modulus, GPa	6.8	2.7
Tensile strength, MPa	54	22
Rupture strain, %	1.6	4.4

where σ_c^N is the critical nominal stress, a is the length of each notch, w is the half specimen width and β is a geometry factor.

The calculated values of K_c are shown in Fig. 1 for MD and in Fig. 2 for CD. It is evident that K_c cannot be a constant material property since it increases rapidly with the notch length.

From Fig. 2 one can also see that the specimen width w has no distinguishable effect on K_c. As seen in Table I, the critical nominal stress in CD is independent of the width at least for $w > 25$ mm. The series expansion of Eq. (1) also shows that β varies very little with the present specimen geometries.

FRACTURE ENERGY

Next we make use of the fracture energy R to calculate K_c. If LEFM is valid, K_c is equal to $\sqrt{RE'}$, where E' is an elastic constant as explained below. We determined the specific essential fracture energy of the copy paper through the short-span tensile test described in Refs. [8], [17] and [22] (cf. [10]). The measured values were 10.2 kJ/m² for MD and 6.5 kJ/m² for CD.

For an isotropic material under plane stress, the elastic constant E' would be equal to the Young's modulus. For anisotropic materials, the appropriate value must be calculated [23,24,25 p. 399]. Assuming that the copy paper is orthotropic, the Poisson ratios are

$$\vartheta_{MD} = 0.25\sqrt{\frac{E_{MD}}{E_{CD}}}$$

and

$$\vartheta_{CD} = 0.25\sqrt{\frac{E_{CD}}{E_{MD}}}$$

[26–30] and the in-plane shear modulus is

$$G = 0.40\sqrt{\frac{E_{MD}}{E_{CD}}}$$

[cf. 26,31], the elastic constants become

$$E'_{MD} = E_{MD}^{3/4} E_{CD}^{1/4}$$

and

$$E'_{CD} = E_{CD}^{3/4} E_{MD}^{1/4}.$$

When the Young's moduli given in Table II are combined with the measured fracture energies, we obtain $K_c = \sqrt{RE'}$ = 7.4 MPa√m in MD and 4.7 MPa√m in CD. These are much higher values than the range of values in Figs. 1 and 2. Likewise, if the value $K_c = \sqrt{RE'}$ is employed to calculate the critical nominal stress from Eq. (1), a gross overestimate is obtained (Fig. 3). We conclude that the standard LEFM does not give self-consistent results for the copy paper specimens studied here.

PLASTICITY CORRECTION

Even though LEFM is not directly applicable to the present case, one might still argue that LEFM is valid if amended by including a correction for the plastic yielding at the crack tip. We can do this by replacing the crack length a in Eq. (1) with $a + \delta a$.

The critical value of the correction δa is sometimes discussed as if it were a material constant. However, the necessary magnitude of this correction depends not only on material properties, but also on the critical value of the noncorrected stress intensity factor [5,6], which in this case

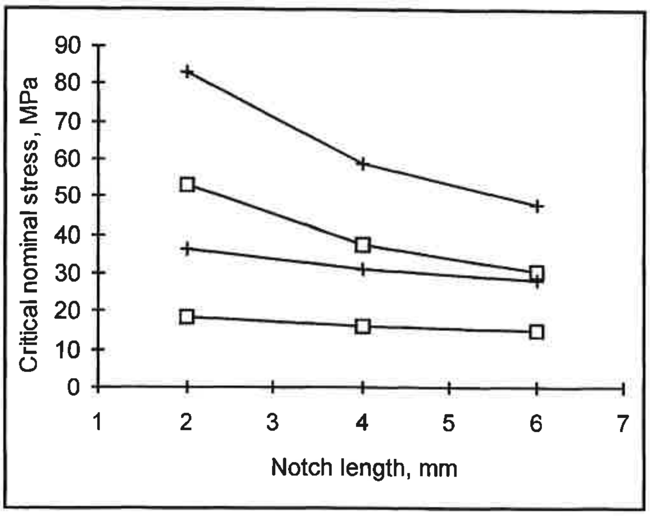

Fig. 3. Critical nominal stress according to the LEFM with $K_c = \sqrt{RE'}$ (top curves) and experiments (bottom curves) for MD (+) and for CD (□).

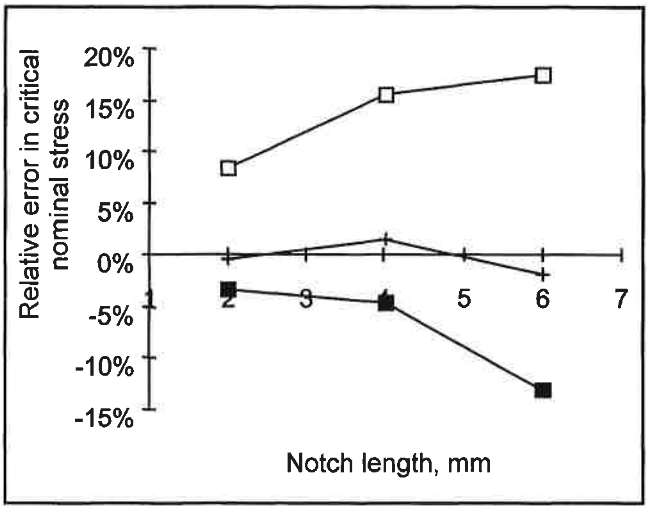

Fig. 4. Relative error in the critical nominal stress calculated from LEFM with the plasticity correction δa = 8.0 mm in MD (+) and 11.7 mm in CD [with w = 25 mm (■) and 50 mm (□)].

significantly varies as a function of notch length (Figs. 1, 2).

If not practically constant for any material, the necessary plasticity correction is a complex function of size, geometry and material properties. Instead of trying to determine this function, we study the consequences of adopting just one plasticity correction value for MD and one for CD.

We choose the values of $\sqrt{RE'}$ determined in the previous section for K_c and then adjust δa so that the error in the critical nominal stress is minimized. This leads to δa = 8.0 mm in MD and 11.7 mm in CD. These values are very large in comparison to notch length and specimen width. Besides, there is a systematic difference between the calculated and measured values, particularly in CD (Fig. 4).

We conclude that using a single value of the plasticity correction for any material direction is not enough to make LEFM agree with the experiments.

COHESIVE SOFTENING MODEL

Strain-softening can be defined as a process where the elastic stiffness of the material decreases when damage is induced by external straining. The specific essential fracture energy can be defined as the energy per crack area which is needed from the onset of softening to the complete separation of the crack faces [cf. 32,33,17]. Typically, at least part of this energy is supplied by the elastic energy that is released when stress decreases due to material damage.

According to a cohesive softening model, there is a stress keeping the crack faces together, even if there is a finite separation between these faces, the separation being defined as the distance between the boundaries of the softening fracture process zone. In the simplest version of the cohesive softening theory, the average cohesive stress between the boundaries is assumed to be a function of the separation only [cf. 34]. This leads to the following definition of the specific essential fracture energy

$$G_{fe} = \int_0^{\delta_c} \sigma(\delta)\mathrm{d}\delta \qquad (2)$$

where G_{fe} is specific essential fracture energy (denoted by R in the more general case above), δ is separation of the zone boundaries (in mm) and σ is cohesive stress. δ_c is critical separation where the cohesive stress is assumed to vanish. In paper applications, the separation δ has been called crack widening [17].

The stress–widening function $\sigma(\delta)$ is assumed to be a material function. It can be determined experimentally using short specimens [17,35]. The measured stress-widening function of the copy paper in MD and CD are shown in Fig. 5.

Apart from LEFM and possibly some other special cases [36,37], no closed-form solution exists that could be used to predict the critical nominal stress from the cohesive softening model. Numerical treatment with a constitutive material model is therefore necessary.

We used an 'elastic-cohesive softening' model [14,16,17,38] in the numerical analysis. The material is assumed to be linearly elastic and orthotropic all the way to the maximum load whereafter it follows the stress-widening function $\sigma(\delta)$. We approximated the $\sigma(\delta)$ shown in Fig. 5 by an exponential function. The specimen geometry and boundary conditions in the two-dimensional plane stress analysis were the same as in the experiments, except that the experiments were conducted under displacement control whereas the analysis was conducted under load control.

The assumed physical process behind the cohesive softening model is as follows. When the double-edge-notched specimen is loaded in tension, a stress concentration appears at the notch tip. Softening begins when the local stress at the tip reaches material strength. In our model this happens already at an infinitesimal external load because, in the absence of softening, the stress would approach infinity close to the notch tips. When the external load is increased, the softened zone or fracture process zone (FPZ) extends further and further away from the original notch tip. Thus the point of highest stress moves away from the original notch tip. At the same time, crack widening takes place at the notch tip and the local stress there decreases because of material damage.

The length of the FPZ at any given level of the external load is determined by the minimum elastic energy in the system. In addition to the material properties, the FPZ length depends on the specimen size and shape. The macroscopic failure takes place when the FPZ keeps on growing even though the external load is decreased. The corresponding load defines the critical nominal stress. For further information about the numerical analysis, see Refs. [14,16,37,38].

The critical nominal stress given by the numerical analysis is shown in Fig. 6 together with the experimental values. In MD, the cohesive softening model overestimates the critical nominal stress up to 5%. This is mostly within the 95% confidence limits of the experimental mean value. In CD, the model yields overestimates of 8–12%. This may arise from the exclusion of plastic yielding in the model. The predicted effect of crack length and specimen geometry on the critical nominal stress agrees with the experiments.

DISCUSSION

We found above that linear elastic fracture mechanics cannot be used to predict the critical nominal stress of specimens with different crack lengths. Even if a fitted value for the plasticity correction δa is used,

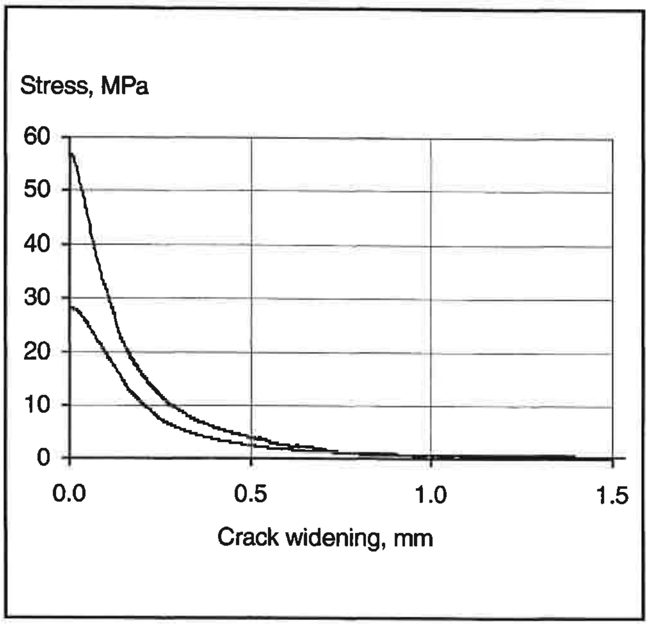

Fig. 5. Cohesive stress–crack widening curves measured for the copy paper for MD (top curve) and CD (bottom).

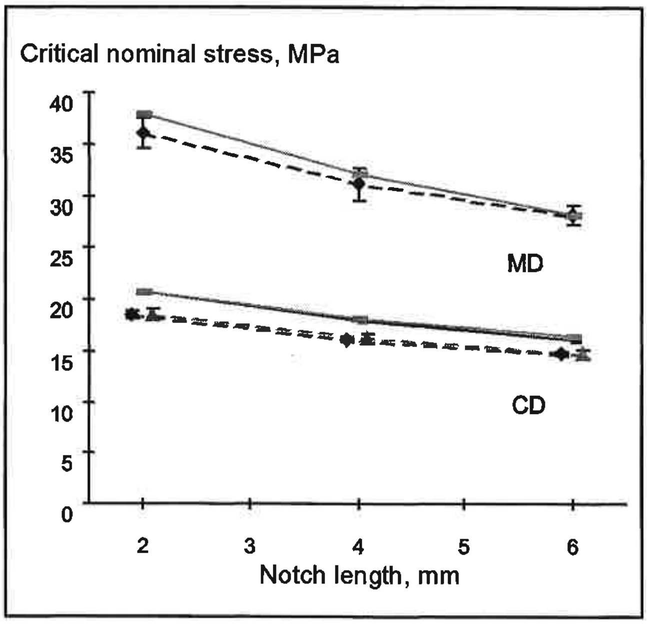

Fig. 6. Critical nominal stress in MD and CD; experiments (- - -) and numerical predictions from the cohesive softening model (—).

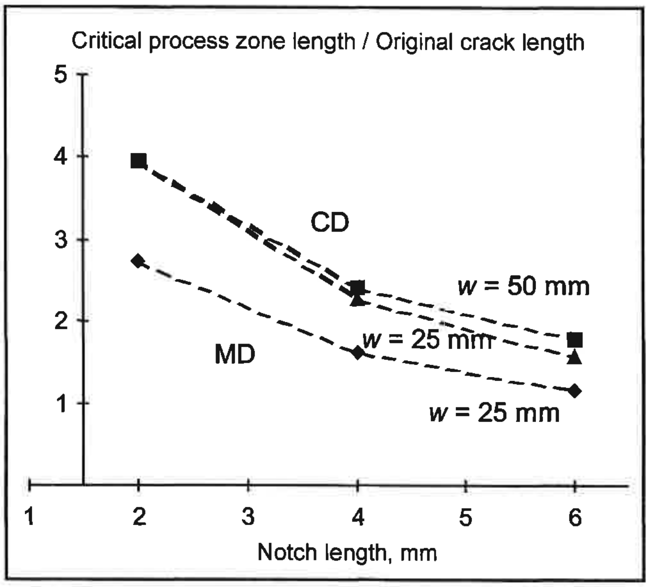

Fig. 7. Critical fracture process zone length in relation to the original notch length according to the cohesive softening model.

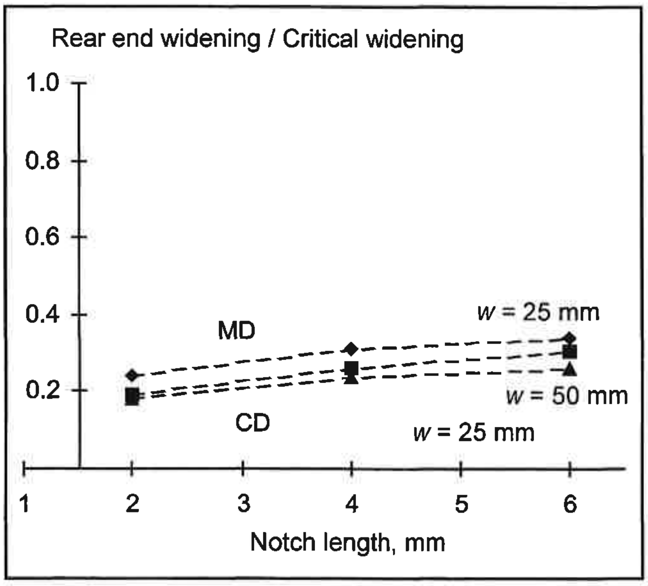

Fig. 8. Crack widening δ at the initial notch tip at the macroscopic instability divided by the value that corresponds to a 95% decrease in the cohesive stress $\sigma(\delta)$.

LEFM cannot be made consistent with the observed effects of crack size and specimen geometry. On the other hand, good agreement is found between experiments and the predictions of a cohesive softening model.

It is interesting to consider why LEFM does not work. There are two questions one should ask (cf. Fig. 3):

1. Why is the critical nominal stress a much weaker function of the notch length than precdicted by LEFM?
2. Why is the critical nominal stress much lower than predicted by LEFM?

In order to answer the first question, it is useful to consider how far from the initial notch tip the fracture process zone has extended at the onset of the macroscopic failure. According to the cohesive softening model, this critical process zone length increases slightly when the initial notch length is increased. However, in relation to the initial notch length, the critical zone length is largest when the initial notch is shortest (Fig. 7). This explains why the relative error in LEFM is largest at the smallest initial notch lengths. LEFM is naturally valid only when the critical process zone length in relation to initial crack length approaches zero.

In order to address the second question, let us study the calculated crack widening at the tip of the original crack at the moment of maximum load in relation to the crack widening where the cohesive stress decreases to 5% of the original value (Fig. 8). We find that, with these structures, the crack widening does not reach the critical value before instability of the structure: macroscopic failure occurs through unstable extension of the front end of the fracture process zone [cf. 14–16]. A consequence is that the entire fracture energy is not "consumed" prior to instability; a significant cohesion between the crack faces still exists at the moment of instability. Thus, the critical nominal stress is much less than as if calculated from LEFM (Fig. 3).

The relative widening of the initial notch tip at the macroscopic instability is smaller in CD than in MD (Fig. 8). This, together with the proportionately larger value of the critical fracture process zone length in CD (Fig. 7), suggests that LEFM should give a larger overestimate of the critical nominal stress in CD than in MD. This is indeed the case: the relative error is 70–130% in MD and 110–190% in CD (Fig. 3).

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