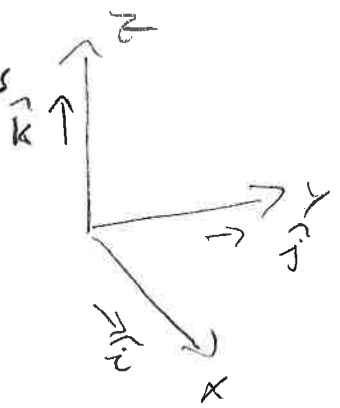
Measurement, Scaling, Instrumentation

Measurement of Volume

1° Stereometric $\begin{cases} \text{Determine some boundaries} \\ \text{Adopt shape assumptions} \\ \text{Compute volume} \end{cases}$

2° Fluid Replacements

3° Buoyancy



Vectorial line elements

$$\ell = \int_A^B \hat{\ell}\cdot d\bar{\ell} = \int_A^B d\ell = \hat{\ell}\cdot\int_A^B d\bar{\ell} = \hat{\ell}\cdot\left[\int_A^C d\bar{i} + \int_C^B d\bar{j}\right]$$

$$= \int_A^C \hat{\ell}\cdot d\bar{i} + \int_C^B \hat{\ell}\cdot d\bar{j}$$

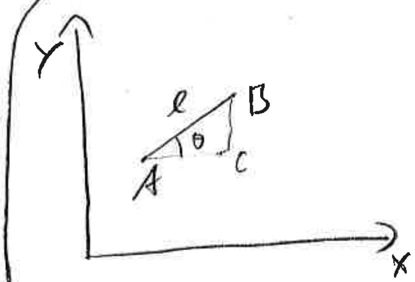
$$= \int_A^C \cos\theta\, dx + \int_C^B \sin\theta\, dy$$

$$= |C-A|\cos\theta + |B-C|\sin\theta$$

$$= |C-A|\,\frac{|C-A|}{|B-A|} + |B-C|\,\frac{|B-C|}{|B-A|} = |B-A|$$

$$|B-A|^2 = |C-A|^2 + |B-C|^2$$

$$d\bar{\ell} = \hat{\ell}\, d\ell$$
$$d\bar{i} = \hat{i}\, dx$$
$$d\bar{j} = \hat{j}\, dy$$
$$d\bar{k} = \hat{k}\, dz$$



$$d\ell^2 = dx^2 + dy^2$$

$$d\ell = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Surface Area Element

$$d\bar{S} = \hat{n}\, dS$$

Surface Area Element in xy-plane

$$d\bar{S}_{xy} = \hat{k}\, dS_{xy} = \hat{k}\left(\hat{k}\cdot d\bar{S}\right) = \hat{k}(\hat{k}\cdot\hat{n})\,dS = \hat{k}\cos\varphi\, dS$$

$$= \hat{k}\, dx\, dy = d\bar{i} \times d\bar{j} = \hat{i}\,dx \times \hat{j}\,dy$$

$$= \hat{k}\, dx\, dy$$

$$\bar{A}\times\bar{B} = A_i B_j \hat{e}_k \varepsilon_{ijk}$$

$$= \begin{bmatrix} e_i & e_j & e_k \\ A_i & A_j & A_k \\ B_i & B_j & B_k \end{bmatrix}$$

$$\hat{i}\times\hat{j} = \begin{bmatrix} \hat{e}_i & \hat{e}_j & \hat{e}_k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \hat{e}_k = \hat{k}$$

Area $A = \int \hat{n}\cdot d\bar{S} = \int dS$

In xy-plane

$$A_{xy} = \int \hat{k}\cdot d\bar{S}_{xy} = \int dS_{xy} = \iint dx\,dy = \int \hat{k}\cdot\hat{n}\,dS = \int \cos\varphi\, dS$$

Volume

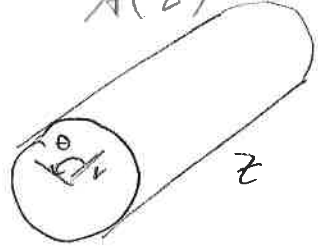
$$V = \int_V dV = \iiint dx\,dy\,dz = \int A_\perp(z)\,dz = \int A_{xy}(z)\,dz = \int \underbrace{A(z)\cos\varphi}_{A(z')}\, dz'$$

$$dz \equiv d(\cos\varphi\, z')$$

In Cylindrical Co-ordinates

$$A = \int_0^{2\pi}\int_0^{r} r'\,dr'\,d\theta = \int_0^{2\pi} \tfrac{1}{2} r^2\, d\theta = \pi r^2$$

If r constant



$$V = \int_0^{\ell} A(z)\,dz$$

Let us compute the volume of a cylinder

$$V = \int_0^{\ell} A(z)\,dz = \int_0^{\ell} \pi r^2\,dz = \pi r^2 \ell$$

And the volume of a cone : $r = az$ $\Rightarrow r_e = a\ell \Rightarrow a = \frac{r_e}{\ell}$

$$V = \int_0^{\ell} A(z)\,dz = \int_0^{\ell} \pi (az)^2\,dz = \pi a^2 \Big|_0^{\ell} \frac{1}{3} z^3 = \frac{\pi a^2 \ell^3}{3} = \frac{\pi r_e^2 \ell}{3}$$

The volume of a frustum of a cone

$r = r_0 + az$

$$V = \int_0^{\ell} A(z)\,dz = \int_0^{\ell} \pi (r_0 + az)^2\,dz = \pi \int_0^{\ell} (r_0^2 + 2r_0 a z + a^2 z^2)\,dz$$

$$= \pi \Big|_0^{\ell} r_0^2 z + r_0 a z^2 + \frac{1}{3} a^2 z^3 = \pi \left[r_0^2 \ell + r_0 a \ell^2 + \frac{1}{3} a^2 \ell^3 \right]$$

What is $r(\ell)$? $r(\ell) = r_0 + a\ell \Rightarrow a = \frac{r_e - r_0}{\ell}$

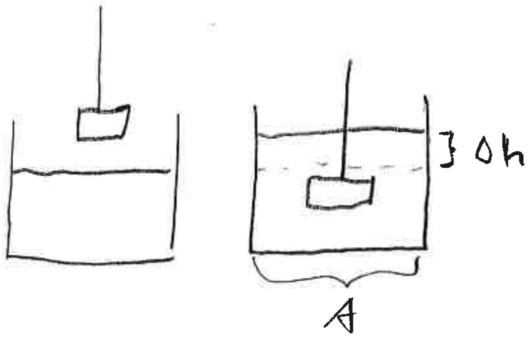
$$V = \pi \ell \left[r_0^2 + r_0 (r_e - r_0) + \frac{1}{3} (r_e - r_0)^2 \right]$$

$$= \frac{\pi \ell}{3} \left[r_e^2 + r_0 r_e + r_0^2 \right] = \frac{\pi \ell}{12} \left[d_0^2 + d_0 d_e + d_e^2 \right]$$

What if our object is not strictly a frustum of a cone?

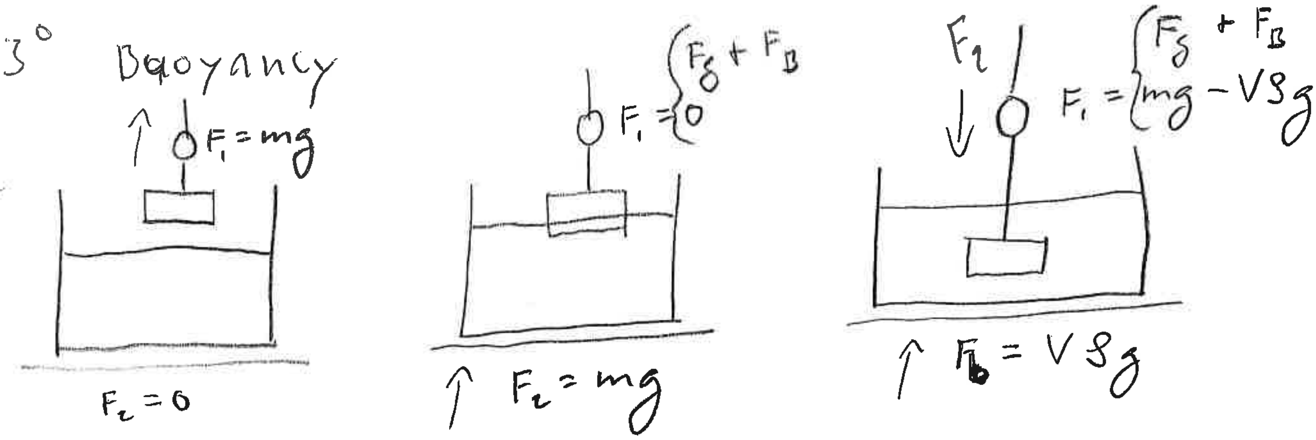
- Within any small section, shape difference will be negligible...
- The same applies even for a cylinder...

2° Fluid Replacement



$$V = A\Delta h$$

3° Buoyancy



$$F_g + F_B = 0$$

$$-|F_g| + F_B = 0$$

$$F_2 + F_g + F_B = 0$$

$$-|F_2| + -|F_g| + F_B = 0$$

$$-|F_2| + -|mg| + \rho g V = 0$$

$$V = \frac{|F_2| + |mg|}{\rho g} = \frac{mg - F_2}{\rho g}$$

$$r(l) = r(0) + \Delta r = r(0) + \int_0^L \frac{dr}{dl}\, dl$$

$$= r(0) + \int_0^L (al)^{\frac{1}{2}}\, dl$$

$$= r_0 + a^{\frac{1}{2}} \Big/_0^L \frac{2}{3} l^{3/2}$$

$$= r_0 + a^{\frac{1}{2}} \frac{2}{3} L^{3/2}$$

$$D(L) = 2r(L)$$

$$A(l) = \pi r^2 = \pi \left[r_0 + a^{\frac{1}{2}} \frac{2}{3} l^{3/2}\right]^2$$

$$V = \int_0^L A(l)\, dl = \pi \int_0^L \left[r_0 + a^{\frac{1}{2}} \frac{2}{3} l^{3/2}\right]^2 dl$$

$$db = dl\sqrt{1+\left(\frac{dr}{dl}\right)^2}$$

$$\frac{db}{dl} = \sqrt{1+\left(\frac{dr}{dl}\right)^2}$$

$$= \sqrt{1+\left[(al)^{\frac{1}{2}}\right]^2}$$

$$\Delta b = \int_0^L \frac{db}{dl}\, dl$$

$$= \int_0^L (1+al)^{\frac{1}{2}}\, dl$$

$$\frac{du}{dl}\, dl = du$$

$$dl = du\, \frac{dl}{du}$$

$$= \int_1^{1+aL} u^{\frac{1}{2}}\, \frac{1}{a}\, du$$

$$= \frac{1}{a} \Big|_1^{1+aL} \frac{2}{3} u^{3/2}$$

$$\frac{d\; 1+al}{dl} = a$$

$$= \frac{1}{a}\, \frac{2}{3}\left[(1+aL)^{3/2} - 1\right]$$

$$dl = \frac{1}{a}\, d(1+al)$$

$$= \frac{1}{a}\, du$$

$$(S_2 - S_1)\,dT = (V_2 - V_1)\,dp$$

$$\frac{dp}{dT} = \frac{S_2 - S_1}{V_2 - V_1} = \frac{\Delta S}{\Delta V} = \frac{\Delta H}{T\,\Delta V}$$

Measurement of mass

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24.3.2021

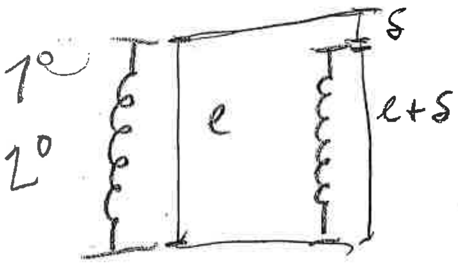
1° Spring displacement

2° Balance of forces

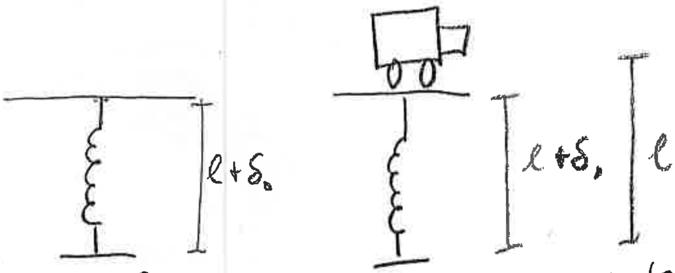
3° Balance of torque momenta

4° Buoyancy - fluid replacement

- others



Spring Equation $F = k\delta$

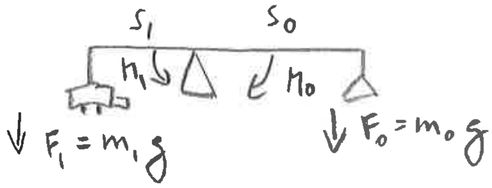


$F_0 = k\delta_0$

$F_1 = k\delta_1 = k(\delta_0 + \Delta\delta_1)$
$= F_0 + \Delta F_1$

$\Delta F_1 = k\,\Delta\delta_1$

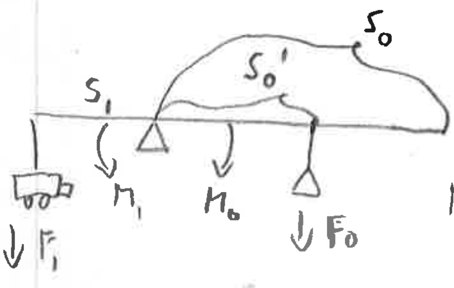
$M_0 + M_1 = 0$



$M_1 + M_0 = 0$

$s_1 m_1 g + s_0 m_0 g = 0$

$m_1 = \frac{-s_0}{s_1} m_0$



$M_1 + M_0 = 0$

$M_1 = s_1 m_1 g + \int_{s_1}^{0} \frac{dm}{ds} s g\, ds$

$M_0 = s_0' m_0 g + \int_{0}^{s_0} \frac{dm}{ds} s g\, ds$

$m_1 = \frac{-s_0' m_0 g - \int_0^{s_0} \frac{dm}{ds} g s\, ds - \int_{s_1}^{0} g \frac{dm}{ds} s\, ds}{s_1 g}$

$= \frac{-s_0' m_0 + \frac{dm}{2ds}\left[s_0^2 - s_1^2\right]}{s_1} = \frac{s_0' m_0 + \frac{dm}{2ds}\left[s_0^2 - s_1^2\right]}{|s_1|}$

Measurement of mass cntd

4°

$F_0 = \rho V g = \rho A h g$

$F_0 + F_1 = 0 \Rightarrow F_1 = -F_0$

$m_1 = \rho A h$

Why $F_0 = -\rho A h g$?

since $F_0 = -\Delta p A = -\rho g \Delta h A$

$V = \Delta h A \Rightarrow F_0 = -\rho g V$

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Surface Energy γA

$$F = \frac{dW}{d\delta} = \frac{d(\gamma A)}{dr} = \frac{d\ \gamma 4\pi r^2}{dr} = 8\pi r \gamma$$

Stress due to surface tension

$$\frac{F}{A} = \frac{8\pi r \gamma}{4\pi r^2} = \frac{2\gamma}{r}$$

Balance of Forces

$$p_{in}\, 4\pi r^2 = p_{out}\, 4\pi r^2 + 8\pi r \gamma$$

$$\Delta p = p_{in} - p_{out} = \frac{2\gamma}{r}$$ Laplace Eq.

Molar Gibbs Function

$$G_m = \frac{G}{n} = \mu N_A$$

$$dG_m = -\frac{S}{n} dT + \frac{V}{n} dp + \frac{\mu}{n} dN$$
$$= -S_m\, dT + V_m\, dp + \mu_m\, dN$$

Equilibrium of liquid w. Ideal Gas

$$\mu_g = \mu_\ell \Rightarrow G_{mg} = G_{m\ell}$$

at $dT = dN = 0$

$$V_{mg}\, dp_g = V_{m\ell}\, dp = V_{m\ell}\left(dp(r=\infty) + d\Delta p\right)$$

Ideal Gas: $pV_m = RT$

$$\frac{RT}{p} dp_g = V_{m\ell}\, dp_\infty + V_{m\ell}\, d\Delta p$$

$$RT \ln p_g = V_{m\ell}\, p_\infty + V_{m\ell}\, \Delta p + C$$
$$= V_{m\ell}\, p_\infty + V_{m\ell} \frac{2\gamma}{r} + C$$

Moisture Content $\frac{m_w}{m_w + m_o}$

Moisture Ratio $\frac{m_w}{m_o}$

Dryness $\frac{m_o}{m_o + m_w}$

Let us discuss some thermodynamic potentials:

Internal Energy

$$U = \underbrace{TS}_{\text{Heat}} \underbrace{-pV}_{\text{work}} \underbrace{+\mu N}_{\text{Chem. pot.}} \qquad dU = TdS - pdV + \mu dN$$

Enthalpy

$$H \equiv U + pV = TS + \mu N \qquad dH = TdS + Vdp + \mu dN$$

Gibb's Function

$$G \equiv U - TS + pV = \mu N \qquad dG = -SdT + Vdp + \mu dN$$

Phase Transition: $dp = dT = 0$

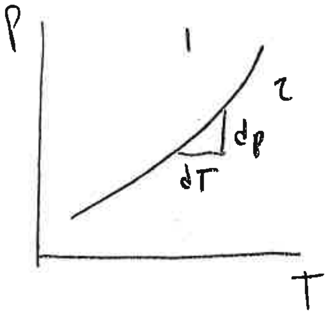
$$\Rightarrow \Delta H = T \Delta S$$

Change of Heat

Coexistence: $dG_1(p, T, N) = dG_2(p, T, N)$

$$\Delta G_2 - \Delta G_1 = 0$$

$$\Delta G_1 = -S_1 dT + V_1 dp$$
$$\Delta G_2 = -S_2 dT + V_2 dp$$



$$-(S_2 - S_1) dT + (V_2 - V_1) dp = 0$$

$$\frac{dp}{dT} = \frac{\Delta S}{\Delta V} = \frac{\Delta H}{T \Delta V}$$

Clausius-Clapeyron Eq.

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$$p_g = e^{\frac{V_{me} p_\infty}{RT}} \, e^{\frac{V_{me} 2\gamma}{rRT}} \, C_2$$

$$= C_3 \, e^{\frac{V_{me} 2\gamma}{rRT}}$$

$$\left\{\begin{array}{l} r \to \infty \quad \Rightarrow \\ e^{\frac{1}{r}} \to 1, \quad p_g \to p_\infty \end{array}\right.$$

$$= p_\infty \, e^{\frac{V_{me} 2\gamma}{rRT}}$$

$$= p_\infty \, e^{\frac{2\gamma M}{rRT\rho}}$$

KELVIN Eq.

Set $p_g (r = r_s) = p_s$

$$p_s = p \, e^{\frac{2\gamma M}{r_s RT\rho}} \quad \Rightarrow \quad p/p_s = e^{-\frac{2\gamma M}{r_s RT\rho}}$$

Clausius-Clapeyron Continued

$$\frac{dp}{dT} = \frac{\Delta H}{T \Delta V}$$

$\Delta V \approx V_{vapor}$

$pV = nRT$

$V_{vapor} \approx \frac{nRT}{p}$

$$\frac{dp}{dT} = \frac{\Delta H p}{nRT^2}$$

$$\frac{dp}{p} = \frac{\Delta H}{nRT^2} dT \quad \Big/ \int$$

$m = n m_{mol}$

$n = \frac{m}{m_{mol}}$

$$\ln p = \frac{-\Delta H}{nRT} + C = -\frac{\Delta H}{m} \frac{m_{mol}}{RT} + C$$

$$p = e^{-\frac{\Delta H}{m}\frac{m_{mol}}{RT}} e^{c} = C_2 e^{-\frac{\Delta H}{m}\frac{m_{mol}}{RT}}$$

$\equiv p_s$ SATURATION VAPOR PRESSURE

For Water:

$\frac{\Delta H}{m} \approx 2260 \ J/g$

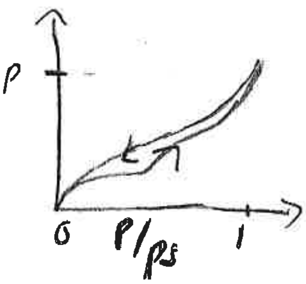
$m_{mol} = 18 \ g/mol$

$R = 8{,}31 \frac{J}{mol\,K}$

Relative Vapor Pressure
Water Activity
Relative Humidity

$\Big\} \ p/p_s \Rightarrow$ Equilibrium moisture content

Hysteresis



Why does MC Increase w. p/p_s ?

KELVIN Eq II :

$$p_r = p_\infty e^{\frac{V 2 \gamma}{RTr}}$$

Vapor pressure in droplet of radius r

$: p_s$

γ = surface tension

V = molar volume

For largest droplet

$$1 = \frac{p}{p_s} e^{\frac{V 2\gamma}{RTr}}$$

$$\frac{p}{p_s} = e^{-\frac{V 2\gamma}{RTr}}$$

$$-\frac{V2\gamma}{RTr} = \ln p/p_s$$

$$r_{max} = -\frac{V 2\gamma}{RT \ln p/p_s}$$

Check the Eq. for $\begin{cases} p \to 0 \\ p \to p_s \end{cases}$

What happens to water activity as a function of Temperature?

$$\frac{\left(p/p_s\right)_2}{\left(p/p_s\right)_1} = \frac{p/p_{s2}}{p/p_{s1}} = \frac{p_{s1}}{p_{s2}} = \frac{e^{-\frac{\Delta H}{m}\frac{m_{mol}}{RT_1}}}{e^{-\frac{\Delta H}{m}\frac{m_{mol}}{RT_2}}} = e^{\frac{\Delta H}{m}\frac{m_{mol}}{R}\left(\frac{1}{T_2}-\frac{1}{T_1}\right)}$$

Say, $\begin{cases} T_1 = 293\ K \\ T_2 = 373\ K \end{cases} \Rightarrow \frac{1}{T_2} - \frac{1}{T_1} = \left(\frac{1}{373} - \frac{1}{293}\right)\frac{1}{K} \approx \left(268 \cdot 10^{-5} - 341 \cdot 10^{-5}\right)\frac{1}{K}$

$$\approx -73 \cdot 10^{-5}\ \frac{1}{K}$$

$$\approx -\frac{1}{1366\ K}$$

$$\frac{\left(p/p_s\right)_2}{\left(p/p_s\right)_1} = e^{-2260\ J/g\ \frac{18\ g/mol}{8{,}31\ \frac{J}{mol\ K}\ 1366\ K}} \approx e^{-3{,}58} \approx 0{,}028$$

Measuremen of RH

- Hygrometers
- Psychrometers
- Dew - Point Sensors
    - Determine Dew Point Temperature, then use Clausius - Clapeyron

$$P/P_s = \frac{e^{-\frac{\Delta H}{m}\frac{m_{mol}}{RT_D}}}{e^{-\frac{\Delta H}{m}\frac{m_{mol}}{RT}}} = e^{-\frac{\Delta H}{m}\frac{m_{mol}}{R}\left(\frac{1}{T_D}-\frac{1}{T}\right)}$$

Measurement of Moisture Content

- Gravimetry
- Distillation
- NMR
- Resistivity / conductivity
- Neutron Moderation / γ-absorption

Determination of Equilibrium Moisture Content

- Determine RH
- Determine MC at equilibrium
  - RH can be arranged using a saturated salt solution at $RH \leq 95\%$
  - For higher humidities, a wet specimen is placed on the top of a porous plate, a known pressure difference arranged to run some water from the specimen through the plate.

    → Water Desorption Isotherm

Which P/P_0 corresponds to Fiber Saturation Point? → Solution from Kelvin Eq

$$P/P_s = e^{-\frac{2 \gamma \eta}{r_s R T \delta}}$$

Determination of FSP through Solute Exclusion Technique

1° Produce a solution of molecules, concentration $c_1 = \frac{m_1}{V_1}$

2° Add wet porous substance, mass of solids in relation to volume of water $c_2 = \frac{m_2}{V_2}$

3° some of the water coming with the substance dilutes the solution, concentration becomes

$$c_3 = \frac{m_1}{V_3}$$

what is now V_3 ?

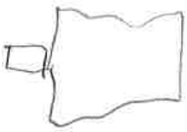
That is water volume accessible to the molecules. $V_1 + V_2 = V_3 + V_4$

V_4 is Inaccessible water volume

$$V_4 = V_1 + V_2 - V_3 = \frac{m_1}{c_1} + \frac{m_2}{c_2} - \frac{m_1}{c_3}$$

$$FSP\left[\frac{(\ )}{(\ )}\right] = \frac{V_4 \cdot \rho_w}{m_2} = \rho_w\left[\frac{m_1}{m_2}\left(\frac{1}{c_1} - \frac{1}{c_3}\right) + \frac{1}{c_2}\right]$$

$V_4 \cdot \rho_w$ = mass of water in pores Inaccessible to molecules



Moisture Content and Electrical Conductivity

Spring Eq.

$$F = K \delta$$

$[N] \quad \left[\frac{N}{m}\right] \quad [m]$

Hooke's Law

$$\frac{F}{A} = E \frac{\delta}{l}$$

$\left[\frac{N}{m^2}\right] \quad \left[\frac{N}{m^2}\right] \quad [\,]$

Potential difference Eq.

$$\Delta Q = R \; I$$

$[V] \quad \left[\frac{V}{A}\right] \quad [A]$

$\left[\frac{J}{C}\right] \quad \left[\frac{J s}{C^2}\right] \quad \left[\frac{C}{s}\right]$

$[\Omega]$

Specific Resistance Eq.

$$\frac{\Delta Q}{s} = \rho \frac{I}{A}$$

$\left[\frac{V}{m}\right] \quad \left[\frac{V}{A} m\right] \quad \left[\frac{A}{m^2}\right]$

$[\Omega m]$

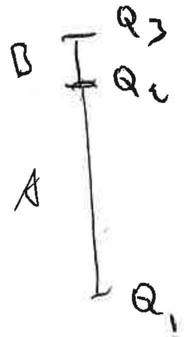
Conductivity Eq.

$$\frac{I}{A} = \frac{1}{\rho} \frac{\Delta Q}{s} = C \frac{\Delta Q}{s}$$

$\left[\frac{A}{m^2}\right] \quad \left[\frac{1}{\Omega m}\right] \quad \left[\frac{V}{m}\right]$

How do we measure conductivity?

Conductors A and B in series



$$I_A = I_B$$

$$\cancel{A}\frac{(Q_3-Q_2)}{S_B} C_B = \cancel{A}\frac{(Q_2-Q_1)}{S_A} C_A$$

$$C_A = \frac{S_A}{S_B}\frac{Q_3-Q_2}{Q_2-Q_1} C_B$$

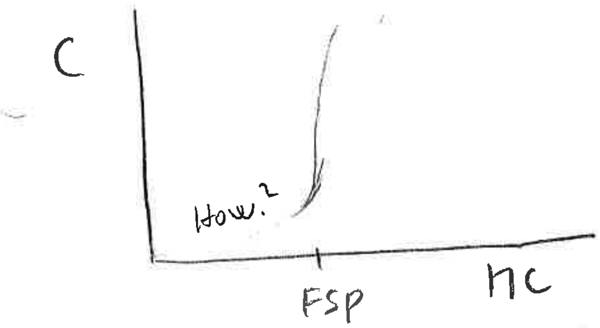
Some Specific Resistances

Dry Wood	10^{12} Ωm	
Distilled Water	$5 \cdot 10^{3}$ Ωm	

$\Big\}$ $c = C(mc)$

$\Rightarrow$ Specific conductivities $10^{-12}\ \frac{1}{\Omega m}$

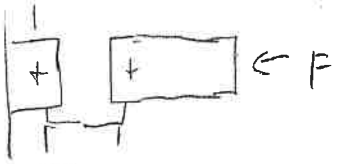
$2 \cdot 10^{-4}\ \frac{1}{\Omega m}$



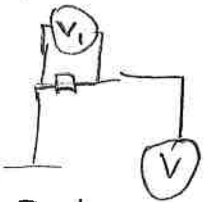
Signal Processing

Signal generator	Displacement due to force
Input transducer	Electrical signal
Signal modifier	Conversion to linear Voltage $\pm 5V$
Output transducer	Conversion to Newtons

1°

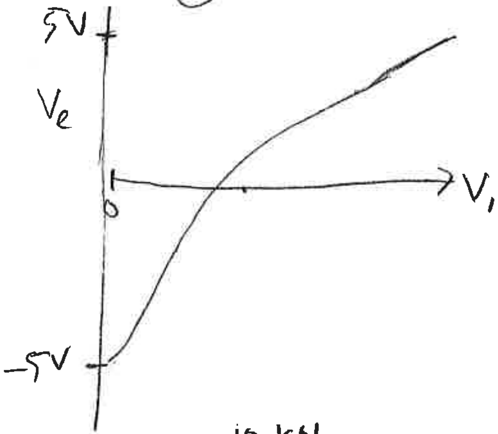


2°

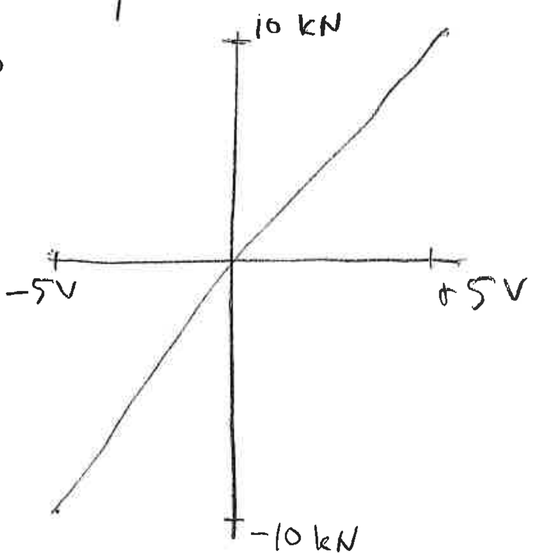


$V_1 = R_1 I$

3°



4°



Eventual problems

Signal generator displacement
→ specimen displacement

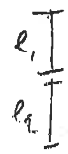
Finite Signal generator stiffness

Finite mounting system stiffness

Signal dampening

Temperature-dependence

Hysteresis



$\delta = \delta_1 + \delta_2$

$= \frac{F}{k_1} + \frac{F}{k_2}$

$F = k_1 \delta_1 = k_2 \delta_2$

Effective K:

$K_e = \frac{F}{\delta} = \frac{F}{\delta_1 + \delta_2}$

$= \frac{F}{\frac{F}{k_1} + \frac{F}{k_2}} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}$

Solutions / Actions

Calibration
- Full range
- Zero offset

Temperature compensation

Signal Generator Displacement Compensation

Noise

Signal-to-Noise-Ratio $\frac{S}{N} = \frac{\text{Signal Amplitude}}{\text{Noise Amplitude}}$

→ Detection limit

Fundamental Noise

- Thermal – thermal movement of charge carriers
- Shot – Impacts by individual charge carriers
- Flicker – (?)

Environmental Noise

- Electric & Magnetic fields
- Radiation
- Mechanical Vibration
- Others...

Car → Heart rate meter

Subway → computer screen

mobile phone →

Solutions - Actions

$\frac{S}{N} \uparrow$

Filtering

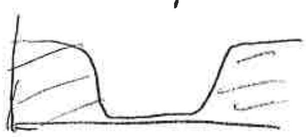
Low-pass

f

High-pass

f

Bandpass



f

Integration
- Boxcar
- Ensemble Averaging
- Moving-Average smoothing
- Weighted Moving-Average smoothing

Dirac δ

$$\int f(t)\,\delta(t-a)\,dt = f(a)$$

$\int \delta(x)\,dx = 1$

$\delta(0) = \infty$

$\delta(t) = 0$ for $t \neq 0$

Fourier Transform

$$\mathfrak{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \;\rightarrow\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{-i\omega u}\, du$$

Inverse Fourier Transform

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathfrak{F}(\omega)\, e^{i\omega t}\, d\omega$$

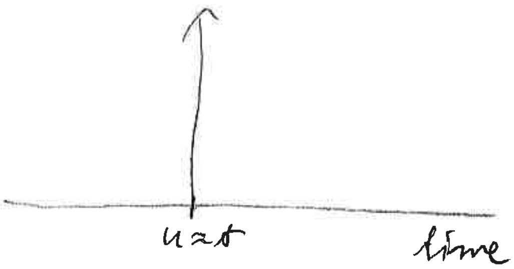
$$= \int_{-\infty}^{\infty} du\; f(u) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t-u)}\, d\omega \right\}$$

$$\Rightarrow \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t-u)}\, d\omega \right\} = \delta(u-t)$$

Euler Identity

$$e^{i\omega(t-u)} = \cos[\omega(t-u)] + i\sin[\omega(t-u)]$$

$\Rightarrow$ ~~Inverse~~ Fourier Transform compresses a whole spectrum of harmonic waves into a narrow range $u \approx t$ in the time domain!



Cumulative Distribution:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}(\varphi)\, e^{i\varphi t}\, d\varphi$$

$$\mathcal{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} d\varphi\, \mathcal{F}(\varphi) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it(\varphi-\omega)}\, dt \right\} \qquad e^{-it(\omega-\varphi)}$$

$$\Rightarrow \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it(\varphi-\omega)}\, dt \right\} = \delta(\varphi - \omega)$$

$\Rightarrow$ (Inverse) Fourier transform compresses a whole spectrum of functions into a narrow
of the time domain
range $\omega \simeq \varphi$ in the frequency domain!

Some Fourier Transformations

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\omega}\,dt = \delta(\omega)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it(\omega-\vartheta)}\,dt = \delta(\omega-\vartheta)$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-it\omega}\,dt = \boxed{\sqrt{2\pi}\,\delta(\omega) = \mathcal{F}(1)}$$

$$f\left[\mathcal{F}(1)\right] = 1$$

$$f\left[\sqrt{2\pi}\,\delta(\omega)\right] = 1 \quad \Rightarrow \quad \boxed{f\left[\delta(\omega)\right] = \frac{1}{\sqrt{2\pi}}}$$

21 b

$$\mathcal{F}\left[e^{itg}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itg} e^{-it\omega}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it(\omega - g)}\, dt = \sqrt{2\pi}\, \delta(\omega - g)$$

$$\mathcal{F}\left[e^{-itg}\right] = \sqrt{2\pi}\, \delta(\omega + g)$$

$$\cos(gt) = \frac{e^{igt} + e^{-igt}}{2}$$

$$\mathcal{F}\left[\cos(gt)\right] = \mathcal{F}\left[\frac{e^{igt} + e^{-igt}}{2}\right] = \frac{\sqrt{\pi}}{\sqrt{2}}\left[\delta(\omega - g) + \delta(\omega + g)\right]$$

Spectroscopy

- Monitoring a spectrum of something
- Affected by interaction with matter
  - Electromagnetic radiation
  - Mechanical waves
  - Particles

Absorption Spectroscopy
Emission Spectroscopy
Scattering Spectroscopy

Photon Energy $E = h\nu = \frac{hc}{\lambda}$
Discrete Energy levels
→ oscillations between states
- several modes of vibration

Classical Wave Equation $\frac{\partial^2}{\partial z^2} A = \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2}$

One Solution : $A = A_0 \cos(kz - \omega t)$

Verification:

$$\frac{\partial}{\partial z} A = -k A_0 \sin(kz - \omega t) \qquad \frac{1}{c^2} \frac{\partial A}{\partial t} = \frac{+\omega}{c^2} A_0 \left[-\sin(kz - \omega t)\right]$$

$$\frac{\partial^2}{\partial z^2} A = -k^2 A_0 \cos(kz - \omega t) \qquad \frac{1}{c} \frac{\partial^2 A}{\partial t^2} = -\frac{\omega^2}{c^2} A_0 \cos(kz - \omega t)$$

$$\Rightarrow \boxed{k^2 = \frac{\omega^2}{c^2}}$$

Intensity $\propto$ (Field density)^2 $= A^2 = A_0^2 \cos^2(kz - \omega t)$

Frequency spectrum of Intensities $A^2(\omega) = \mathcal{F}(A^2(t))$

Lognitudinal Mechanical Waves

$I = F\Delta t$

$P = mv = m\frac{\Delta s}{\Delta t}$

Fluid in a tube

Momentum - Impulse theorem

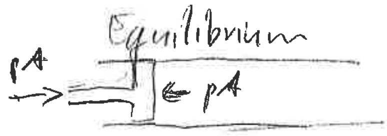
$F\Delta t = m\frac{\Delta v}{\Delta t}\Delta t$

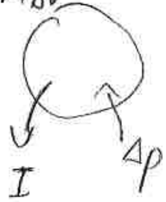
$\Delta P = m\Delta v$

$$\Delta P + I = 0$$

$$\Delta P = -I$$

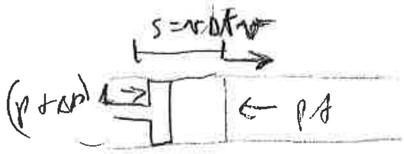
Equilibrium





Additional pressure Δp applied for time Δt

$\Rightarrow$ Acceleration of a Fluid Element of length $s = v\Delta t$, v = velocity of the frontier of the element



v_z = velocity of the piston $\approx$ velocity of the element

Impulse by the Fluid element: $I = -A\Delta p\,\Delta t$

Momentum change by the Fluid element: $\Delta P = (\rho v\Delta t A)\, v_z$

Fractional volume change of the fluid element: $\frac{\Delta V}{V} = -\frac{v_z\Delta t A}{v\Delta t A} = -\frac{v_z}{v}$

Bulk modulus $\equiv \frac{-\Delta p}{\Delta V / V} = B$

$\Rightarrow \Delta p = -B\frac{\Delta V}{V} = B\frac{v_z}{v}$

$$\Delta P = -I$$

$$\rho v\Delta t A v_z = A\Delta p\,\Delta t$$

$$\rho v v_z = B\frac{v_z}{v}$$

$$v^2 = \frac{B}{\rho} \Rightarrow \boxed{v = \sqrt{\frac{B}{\rho}}}$$

Intensity of longitudinal wave

Differential work $\quad dW = \frac{\partial W}{\partial x_1} dx_1 + \frac{\partial W}{\partial x_2} dx_2 + \frac{\partial W}{\partial x_3} dx_3$

$$= F_1\, dx_1 + F_2\, dx_2 + F_3\, dx_3$$

$$= F_{i'}\, dx_{i'}$$

Force in one dimension

$$\rightarrow \quad dW = F_1 dx_1 = F dx$$

Power $\quad \frac{dW}{dt} = \frac{F dx}{dt} = F\frac{dx}{dt} = Fv$

$$\rightarrow Fv_z$$

$$\frac{dW}{dt} = Fv_z = \Delta p A v_z = B\frac{v_z}{v} A v_z$$

$$= B\frac{v_z^2}{v} A$$

$$v_z \equiv \frac{d\Delta}{dt}$$

$$\text{Intensity} = \frac{\text{Power}}{\text{Area}} = \frac{dW}{A\,dt} = B\frac{v_z^2}{v} = B\left(\frac{d\Delta}{dt}\right)^2 \frac{1}{\sqrt{\frac{B}{\rho}}} = \sqrt{B\rho}\left(\frac{d\Delta}{dt}\right)^2$$

Say $\quad \Delta(x,t) = A_\Delta \sin(\omega t - kz) = A_\Delta \sin \omega\left(t - \frac{z}{v}\right)$

$$\omega = 2\pi f = \frac{2\pi}{\tau}$$

$$k = \frac{\omega}{v} = \frac{2\pi}{\lambda}$$

$$\frac{d\Delta}{dt} = A_\Delta\, \omega \cos(\omega t - kz)$$

$$\left(\frac{d\Delta}{dt}\right)^2 = A_\Delta^2\, \omega^2 \cos^2(\omega t - kz)$$

$$\text{Intensity} = \sqrt{B\rho}\, A_\Delta^2\, \omega^2 \cos^2(\omega t - kz)$$

Cycle
Average Intensity $= \frac{1}{\tau}\int_0^\tau \sqrt{B\rho}\, A_\Delta^2\, \omega^2 \cos^2(\omega t - kz)\, dt = \frac{1}{2}\sqrt{B\rho}\, A_\Delta^2 \omega^2$

$$= \frac{1}{2}\rho v A_\Delta^2 \omega^2$$

How to determine A_Δ ?

$$\Delta p = B \frac{v_z}{v} = B \frac{d\Delta}{dt} \frac{1}{v}$$

If $\frac{d\Delta}{dt} = A_\Delta \omega \cos(\omega t - kz)$

$$\Delta p = \frac{B}{v} A_\Delta \omega \cos(\omega t - kz) = A_p \cos(\omega t - kz)$$

$$\Rightarrow A_p = \frac{B}{v} A_\Delta \omega \Rightarrow A_\Delta = A_p \frac{v}{B\omega} = A_p \frac{1}{\sqrt{B\varsigma}\,\omega}$$

→ Measure pressure amplitude, compute displacement amplitude!

Thermal transitions

- changes in thermal properties

First-order transition

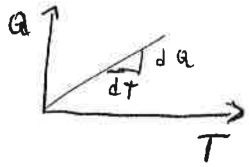
- change in heat capacity
- latent heat involved

Second-order transition

- change in heat capacity only

Heat: thermal energy [J] Q

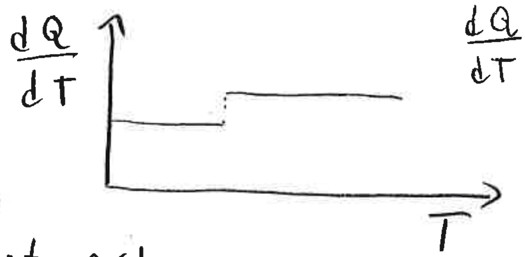
Heat Capacity: $\frac{dQ}{dT}$



Heat flow rate $\frac{dQ}{dt}$

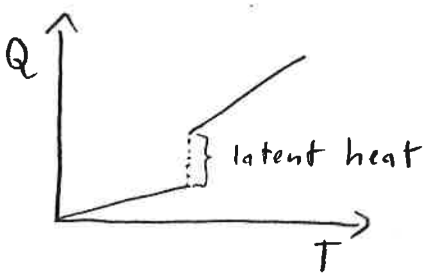
Temperature change rate $\frac{dT}{dt}$

Thermal transition

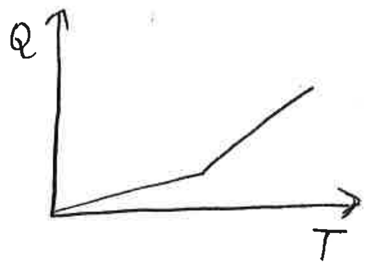


$$\frac{dQ}{dT} = \frac{dQ/dt}{dT/dt}$$

First-order



Second-order



Let us produce heat in a resistor:

Potential difference $\Delta P = P_2 - P_1$

$[V] = \left[\frac{J}{C}\right]$

Current I

$[A] = \left[\frac{C}{s}\right]$

Power $\Delta P\, I \rightarrow \boxed{\frac{dQ}{dt}}$

$\left[\frac{J}{s}\right]$

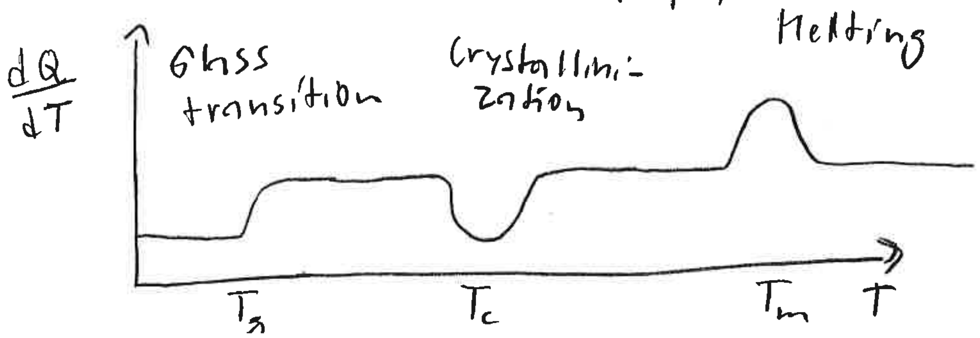
Heat energy $\int \Delta P\, I\, dt$

$\rightarrow$ dissipated as heat

Calorimetry:

Measurement of Heat flows

Thermal transitions of polymers



Melting Temperature Spectrum

Coexistence of Solid and Liquid

Chemical potential $\mu = \frac{G}{N}$ must be equal

$$d\mu^s = d\mu^l$$

$$dG^s = dG^l$$

$$-S^s\,dT + V^s\,dp^s = -S^l\,dT + V^l\,dp^l$$

$$(S^s - S^l)\,dT = V^s\,dp^s - V^l\,dp^l$$

$$-\Delta S\,dT = V^s\,d(p^l + \Delta p) - V^l\,dp^l$$

$$-\frac{\Delta H\,dT}{T} = (V^s - V^l)\,dp^l + V^s\,d(\Delta p) \qquad \Delta H = T\Delta S$$

$$\approx V^s\,d(\Delta p) \qquad \Delta S = \frac{\Delta H}{T}$$

$$\frac{dT}{T} = \frac{-V^s}{\Delta H}\,d(\Delta p) \quad \Big| \int$$

$$\ln T = -\frac{V^s}{\Delta H}\Delta p + C = -\frac{V^s}{\Delta H}\frac{2\gamma}{r} + C$$

$$= -\frac{V^s}{\Delta H}\frac{2\gamma}{r} + \ln T_0$$

$$\ln T - \ln T_0 = \ln\frac{T}{T_0} = -\frac{V^s}{\Delta H}\frac{2\gamma}{r}$$

$$r_m = -\frac{V^s}{\Delta H}\frac{2\gamma}{\ln\frac{T_m}{T_0}}$$

Melting Temperature Spectrum
$\rightarrow$ pore size distribution

T [°C]	D [nm]
-30	1,4
-10	4,2
-5	8,6
-2,5	17
-1,2	36
-0,6	72
-0,3	144
-0,2	216
-0,1	433

Mass of freezing water in pores of diameter between D_1 and D_2

$$\left[m_{FW}\right]_{D_1 D_2} = \frac{1}{\Delta H_m} \int_{T_1(D_1)}^{T_2(D_2)} \frac{dQ_e}{dT}\, dT = \frac{1}{\Delta H_m} \int_{t_1(T_1)}^{t_2(T_2)} \frac{dQ_e}{dt}\, dt$$

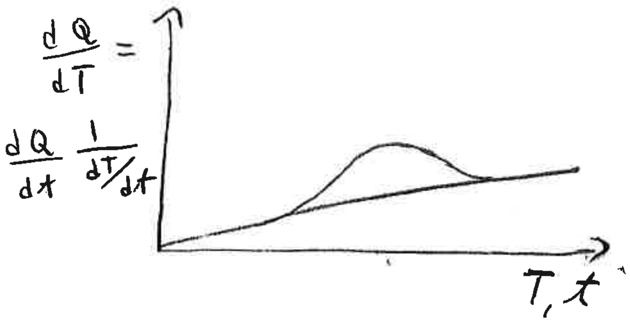
$$\Delta H_m \simeq 333\ \text{J/g}$$

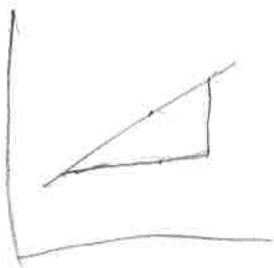
How about the effect of Heat Capacity?

$$Q = Q_T(\text{transition}) + Q_S(\text{temperature increment}) \Rightarrow Q_T = Q - Q_S$$

$$\left[m_{FW}\right]_{D_1 D_2} = \frac{1}{\Delta H_m} \int_{T_1(D_1)}^{T_2(D_2)} \left(\frac{dQ}{dT} - \frac{dQ_S}{dT}\right) dT \approx \frac{1}{\Delta H_m} \left[\int_{T_1}^{T_2} \frac{dQ}{dT}\, dT - \frac{dQ_S}{dT}(T_2 - T_1)\right]$$

How to determine Heat Capacity $\frac{dQ_S}{dT}(T)$?





Non-Freezing Water

$$NFW = \left[m_W - \left[m_{FW}\right]_{-\infty}\right] \frac{1}{m_0}$$

Total Cell Wall Water

$$m_{CW} = NFW + \left[m_{FW}\right]_{-\infty\, T_0^-} \simeq FSP \cdot m_0\ (?)$$

Internal Energy

$$U = U(S, V, N)$$

Energy $\qquad dE = \underset{\text{Heat}}{dQ} - \underset{\text{Work}}{dW}$

Sign convention:

$$dU = dE + \mu dN$$

$$= dQ - dW + \mu dN$$

$$dU = \left(\frac{\partial U}{\partial S}\right)_{V,N} dS + \left(\frac{\partial U}{\partial V}\right)_{S,N} dV + \left(\frac{\partial U}{\partial N}\right)_{S,V} dN$$

Chain Rule of partial derivatives
$\rightarrow$ Total Differential

$$dU = T dS - p dV + \mu dN$$

$$T = \left(\frac{\partial U}{\partial S}\right)_{V,N}$$

$$-p = \left(\frac{\partial U}{\partial V}\right)_{S,N}$$

$$\mu = \left(\frac{\partial U}{\partial N}\right)_{S,V}$$

Maxwell Relations (some)

$$dE = T\,dS - p\,dV$$

$$T = \left(\frac{\partial E}{\partial S}\right)_V \quad \Big| \ \frac{\partial}{\partial V} \quad \Rightarrow \quad \frac{\partial^2 E}{\partial V \partial S} = \left(\frac{\partial T}{\partial V}\right)_S$$

$$-p = \left(\frac{\partial E}{\partial V}\right)_S \quad \Big| \ \frac{\partial}{\partial S} \quad \Rightarrow \quad \frac{\partial^2 E}{\partial S \partial V} = -\left(\frac{\partial p}{\partial S}\right)_V$$

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V \qquad \boxed{(S,V)\text{ - system}}$$

Gibbs Function

$$G \equiv U - TS + pV$$

$$dG = dU - T\,dS - S\,dT + p\,dV + V\,dp = -S\,dT + V\,dp + \mu\,dN$$

$$S = -\left(\frac{\partial G}{\partial T}\right)_{p,N} \quad \Big| \ \frac{\partial}{\partial p} \quad \Rightarrow \quad -\frac{\partial^2 G}{\partial p \partial T} = \left(\frac{\partial S}{\partial p}\right)_{T,N}$$

$$V = \left(\frac{\partial G}{\partial p}\right)_{T,N} \quad \Big| \ \frac{\partial}{\partial T} \quad \Rightarrow \quad \frac{\partial^2 G}{\partial T \partial p} = \left(\frac{\partial V}{\partial T}\right)_{p,N}$$

$$\left(\frac{\partial S}{\partial p}\right)_{T,N} = -\left(\frac{\partial V}{\partial T}\right)_{p,N} \qquad \boxed{(T,p,N)\text{ - system}}$$

Kelvin's Thermoelastic Eq.

$$dQ = T\,dS$$

$$dS = \left(\frac{\partial S}{\partial T}\right)_p dT + \left(\frac{\partial S}{\partial p}\right)_T dp$$

$\boxed{(T,p)\text{-system}}$

Isothermal process $\Rightarrow dT = 0$

$$dQ = T\,dS = T\left(\frac{\partial S}{\partial p}\right)_T dp = -T\left(\frac{\partial V}{\partial T}\right)_p dp$$

$$dp = -\frac{\Delta F}{A}$$

$$\partial V = \Delta V = A\,\Delta \ell$$ Uniaxial elongation

$$\frac{\partial \ell/\ell}{\partial T} = \frac{\Delta \ell/\ell}{\Delta T} = \alpha$$ Linear Thermal expansion coefficient

$$\frac{1}{\Delta T} = \alpha\frac{\ell}{\Delta \ell}$$

$$\Delta Q = -T\,A\,\Delta\ell\,\alpha\frac{\ell}{\Delta\ell}\left(-\frac{\Delta F}{A}\right) = T\alpha\ell\,\Delta F$$

$$\boxed{\frac{\Delta Q}{\Delta F} = T\alpha\ell}$$

$$\boxed{\frac{\Delta Q/V}{\Delta\sigma} = T\alpha}$$

$$\left.\begin{aligned}\Delta\sigma &= \frac{\Delta F}{A}\\ V &= A\ell\end{aligned}\right\}$$

Energy Change in Thermoelastic Uniaxial Straining

$$dE = dQ - dW$$

$$= T\alpha\ell \, dF + F \, d\delta$$

$$F\,d\delta = F\ell \, d\varepsilon$$
$$= A\ell\sigma \, d\varepsilon$$
$$= VY\varepsilon \, d\varepsilon$$

$$\frac{dF/A}{d\delta/\ell} = \frac{d\sigma}{d\varepsilon} \equiv Y$$

$$dF = YA \, d\varepsilon$$

$$dE = T\alpha VY \, d\varepsilon + VY\varepsilon \, d\varepsilon$$

$$= VY\left[T\alpha \, d\varepsilon + \varepsilon \, d\varepsilon\right] \qquad \Big| \int$$

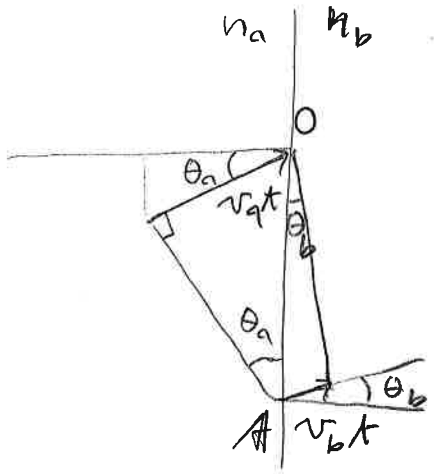
$$\Delta E = VY\left[\underset{\text{Heat}}{T\alpha\varepsilon} + \underset{\text{Work}}{\tfrac{1}{2}\varepsilon^2}\right] \qquad \Big| \frac{d}{dt}$$

$$\dot{E} = VY\left[T\alpha\dot{\varepsilon} + \varepsilon\dot{\varepsilon}\right]$$

What if there is an Irreversible process?

$$dU \le TdS - pdV + \mu dN$$

Snell's Law of Refraction



$$\sin\theta_a = \frac{v_a t}{AO} \Rightarrow AO = \frac{v_a t}{\sin\theta_a}$$

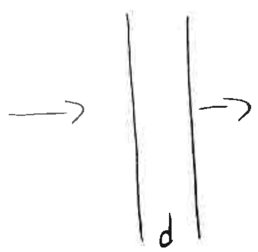
$$\sin\theta_b = \frac{v_b t}{AO} \Rightarrow AO = \frac{v_b t}{\sin\theta_b}$$

$$\frac{\sin\theta_a}{\sin\theta_b} = \frac{v_a}{v_b} = \frac{c/v_b}{c/v_a} = \frac{n_b}{n_a}$$

Birefringence: Anisotropic Index of Refraction

$$n_x = c/v_x \neq n_y = \frac{c}{v_y}$$

Passage through a layer d



Passage Time difference $\quad t = \frac{d}{v}$

$$\Delta t = d\left(\frac{1}{v_x} - \frac{1}{v_y}\right)$$

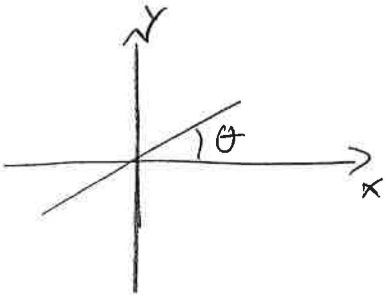
Path Difference (after crossing the layer)

$$P = \Delta t\, c = d\left(\frac{c}{v_x} - \frac{c}{v_y}\right) = d\,(n_x - n_y)$$

Phase Difference

$$\delta = 2\pi \frac{P}{\lambda} = 2\pi \frac{d}{\lambda} \underbrace{(n_x - n_y)}_{\text{Birefringence}}$$

Apply linearly polarized light

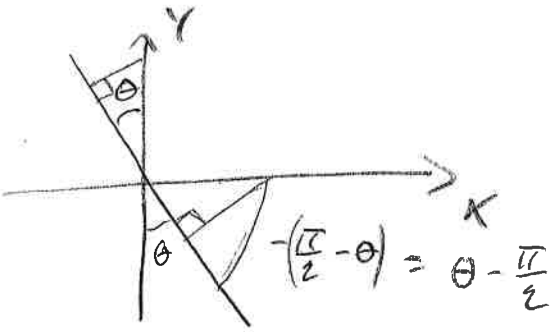


$$A = A_0 \sin \omega t$$
$$A_x = A \cos\theta = A_0 \cos\theta \sin(\omega t)$$
$$A_y = A \sin\theta = A_0 \sin\theta \sin(\omega t)$$

After passage of layer d

$$A'_x = A_0 \cos\theta \sin(\omega t)$$
$$A'_y = A_0 \sin\theta \sin(\omega t + \delta)$$

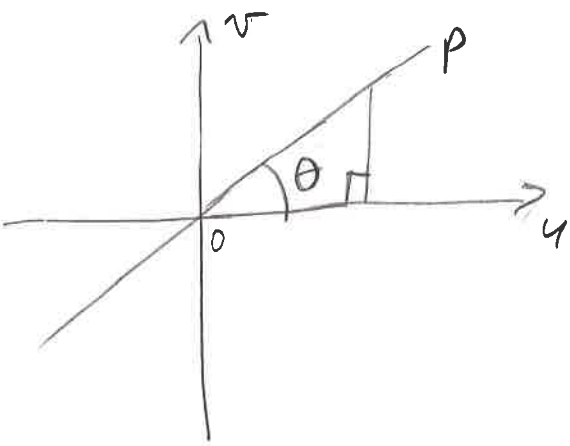
Projections to Analyzer, perpendicular to polarizer



$$S = \cos\theta A'_y - \cos\left(\theta - \frac{\pi}{2}\right) A'_x$$
$$= \cos\theta \sin\theta \sin(\omega t + \delta) A_0$$
$$- \sin\theta \cos\theta \sin(\omega t) A_0$$

$$= \sin\theta\cos\theta \left[\sin(\omega t + \delta) - \sin(\omega t)\right] A_0$$
$$= \frac{A_0}{2} \sin(2\theta) \left[\sin(\omega t + \delta) - \sin(\omega t)\right]$$
$$\left[\sin\left(\omega t + 2\pi \frac{P}{\lambda}\right) - \sin(\omega t)\right]$$
$$\left[\sin\left[\omega t + 2\pi \frac{d}{\lambda}(n_x - n_y)\right] - \sin(\omega t)\right]$$

Elliptical Polarization



Apply linearly polarized light (along OP)

$$A = A_0 \sin \omega t$$

$$\Rightarrow A_u = A \cos\theta = A_0 \sin\omega t \cos\theta$$

$$A_v = A \sin\theta = A_0 \sin\omega t \sin\theta$$

Passing through birefringent material induces phase retardation

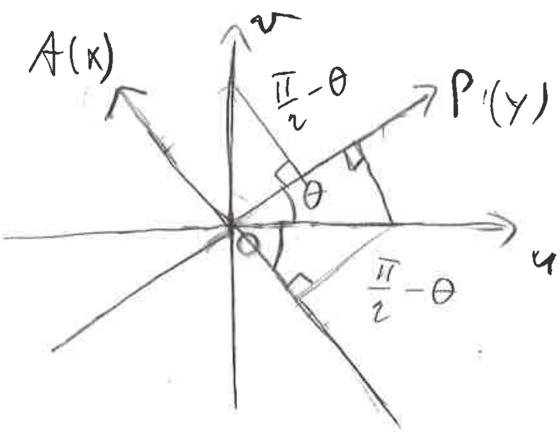
$$A'_u = A_0 \cos\theta \sin(\omega t - \delta) \rightarrow \sin(\omega t)$$

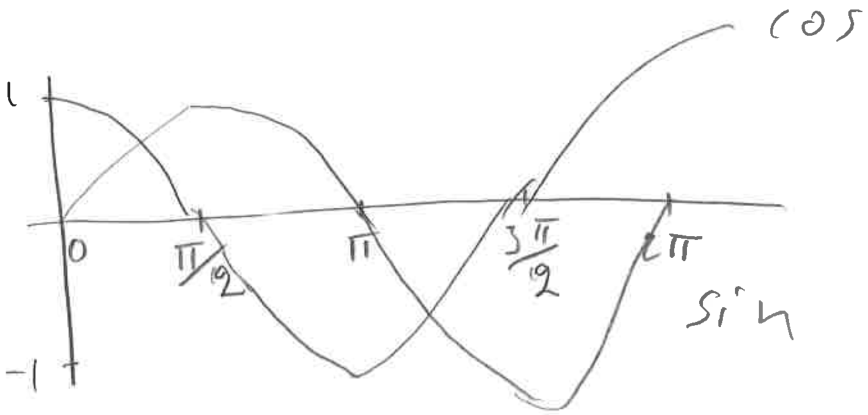
$$A'_v = A_0 \sin\theta \sin(\omega t) \rightarrow \sin(\omega t + \delta)$$

Project into (P, A) - crd-system

$$A_y = A'_u \overset{\cos\theta}{\sin\left(\frac{\pi}{2} - \theta\right)} + A'_v \overset{\sin\theta}{\cos\left(\frac{\pi}{2} - \theta\right)}$$

$$A_x = -A'_u \underbrace{\cos\left(\frac{\pi}{2} - \theta\right)}_{\sin(\theta)} + A'_v \underbrace{\sin\left(\frac{\pi}{2} - \theta\right)}_{\cos\theta}$$





$$\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)$$

$$\sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = -\sin\left(\theta - \frac{\pi}{2}\right) = \cos(\theta)$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$$

$$A_y = A_u \cos(\theta) + A'_v \sin(\theta)$$

$$A_x = -A'_u \sin(\theta) + A'_v \cos(\theta)$$

$$A_y = A_0\left[\cos^2\theta \sin(\omega t - \delta) + \sin^2\theta \sin(\omega t)\right]$$

$$A_x = A_0\left[-\sin(\theta)\cos(\theta) \sin(\omega t - \delta) + \sin(\theta)\cos(\theta) \sin(\omega t)\right]$$

Some Trigonometry

$$\sin(a-b) = \sin(a)\cos(b) - \sin(b)\cos(a)$$

$$\sin(2a) = 2\sin(a)\cos(a)$$

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

$$\cos(2a) = 2\cos^2 a - 1 = 1 - 2\sin^2 a$$

$$1 - \cos(2a) = 2\sin^2 a$$

A Special Case: $\theta = \frac{\pi}{4}$

$$\Rightarrow \sin(\theta) = \cos(\theta) = \frac{1}{\sqrt{2}}$$

$$A_y = \frac{A_0}{2}\left[\sin(\omega t) + \sin(\omega t - \delta)\right]$$

$$A_x = \frac{A_0}{2}\left[\sin(\omega t) - \sin(\omega t - \delta)\right]$$

$$A_y = \frac{A_0}{2}\left[\sin(\omega t) + \sin(\omega t)\cos(\delta) - \sin(\delta)\cos(\omega t)\right]$$

$$= \frac{A_0}{2}\left[\sin(\omega t)\left(1 + \cos\delta\right) - \cos(\omega t)\, 2\sin\left(\frac{\delta}{2}\right)\cos\left(\frac{\delta}{2}\right)\right]$$

$$= \frac{A_0}{2}\left[\sin(\omega t)\, 2\cos^2\left(\frac{\delta}{2}\right) - \cos(\omega t)\, 2\sin\left(\frac{\delta}{2}\right)\cos\left(\frac{\delta}{2}\right)\right]$$

$$= A_0\left[\sin(\omega t)\cos\left(\frac{\delta}{2}\right) - \cos(\omega t)\sin\left(\frac{\delta}{2}\right)\right]\cos\left(\frac{\delta}{2}\right)$$

$$= A_0 \sin\left(\omega t - \frac{\delta}{2}\right)\cos\left(\frac{\delta}{2}\right)$$

$$A_x = \frac{A_0}{2}\left[\sin(\omega t) - \sin(\omega t - \delta)\right]$$

$$= \frac{A_0}{2}\left[\sin(\omega t) - \sin(\omega t)\cos(\delta) + \sin(\delta)\cos(\omega t)\right]$$

$$= \frac{A_0}{2}\left[\sin(\omega t)(1-\cos\delta) + 2\sin\left(\frac{\delta}{2}\right)\cos\left(\frac{\delta}{2}\right)\cos(\omega t)\right]$$

$$= \frac{A_0}{2}\left[\sin(\omega t)\, 2\sin^2\left(\delta/2\right) + 2\sin\left(\frac{\delta}{2}\right)\cos\left(\frac{\delta}{2}\right)\cos(\omega t)\right]$$

$$= A_0\left[\sin(\omega t)\sin\left(\frac{\delta}{2}\right) + \cos(\omega t)\cos\left(\frac{\delta}{2}\right)\right]\sin\left(\delta/2\right)$$

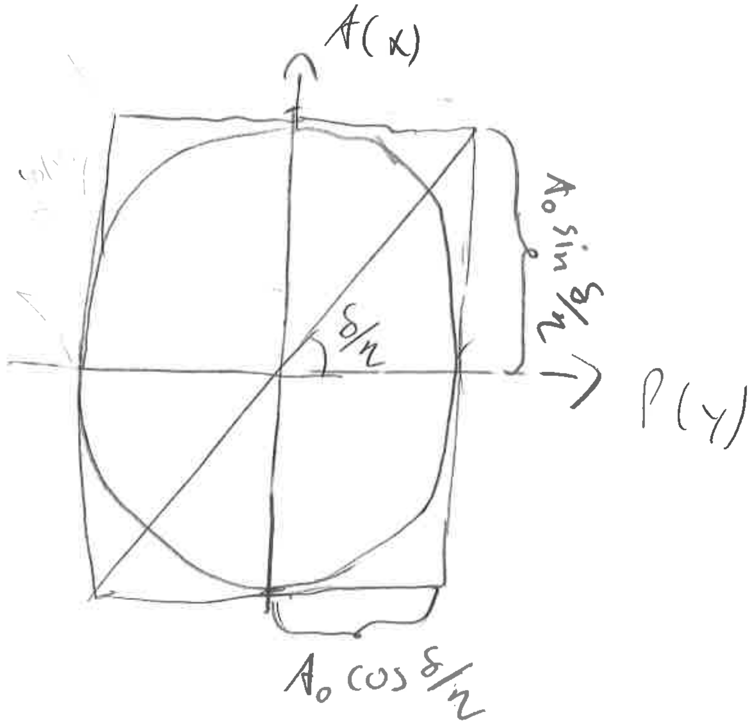
$$= A_0 \cos\left(\omega t - \delta/2\right)\sin\left(\delta/2\right)$$

Square, normalize and add.

$$\frac{A_x^2}{A_0^2\sin^2(\delta/2)} + \frac{A_y^2}{A_0^2\cos^2(\delta/2)} = \cos^2\left(\omega t - \delta/2\right) + \sin^2\left(\omega t - \delta/2\right)$$

$$= 1$$

ELLIPTICAL POLARIZATION!



Rewrite still once:

$$A_x = A_0 \sin\left(\frac{\delta}{2}\right) \cos\left(\omega t - \frac{\delta}{2}\right)$$

$$= A_0 \sin \delta/2 \ \sin\left(\omega t - \delta/2 + \pi/2\right)$$

$$A_y = A_0 \cos \delta/2 \ \sin\left(\omega t - \delta/2\right)$$

$\Rightarrow$ There always is an (x, y)-phase difference of $\pi/2$! $\left(\theta = \frac{\pi}{4}\right)$

Could we get rid of it?

If yes, we would have

a linearly polarized wave in the direction $\vartheta = \delta/2$

i.e. $$\frac{A_x}{A_y} = \tan \delta/2 = \tan \vartheta$$

This Angle can be determined by turning the Analyzer to distinction:

$$\delta = 2\vartheta = 2\pi \frac{P}{\lambda} = 2\pi \frac{d}{\lambda}(n_u - n_v)$$

How?

Use a Quater Wave Plate ...

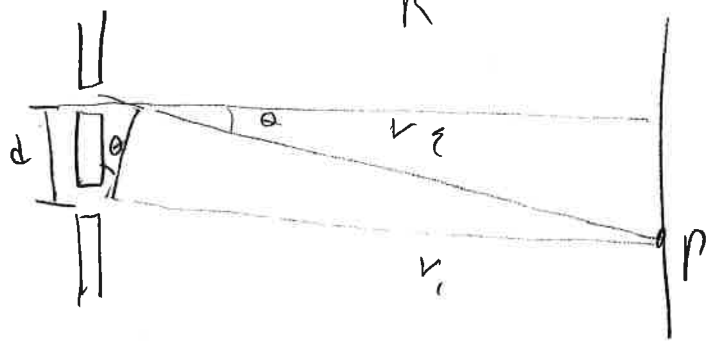
$\rightarrow$ de Sénarmont Compensator

27.4.2021

$$A_x = A_0 \sin\theta \cos\theta \left[\sin(\omega t + \delta) - \sin(\omega t) \right]$$

$$A_y = A_0 \left[\sin^2\theta \sin(\omega t + \delta) + \cos^2\theta \sin(\omega t) \right]$$

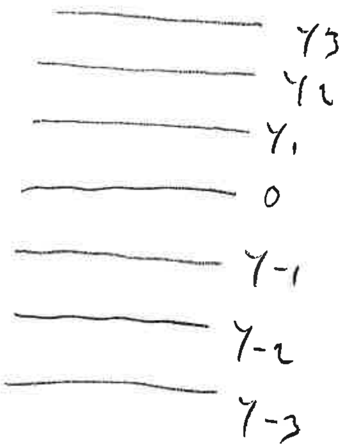
Interference → Diffraction



Positive Interference: $d \sin\theta = m\lambda$

Destructive Interference: $d \sin\theta = \left(m + \frac{1}{2}\right)\lambda$ $m = 0, \pm 1, \pm 2, \ldots$

Interference Pattern:



$$\frac{y_m}{R} = \tan\theta_m \approx \sin\theta_m$$

$$y_m \approx R \sin\theta_m = R \frac{m\lambda}{d}$$