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SOME THEORETICAL CONSIDERATIONS ON THE MECHANICAL PROPERTIES OF FIBROUS STRUCTURES

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Synopsis

Mathematical theories for some of the mechanical properties of a well-consolidated anisotropic fibrous web are developed in two major divisions: the **elastic** *and the* **plastic** *regimes of stress/strain relationships. For each of the principal directions in the plane of the sheet, theories of the* **elastic** *regime are developed for external load, Poisson's ratio, Young's modulus and the modulus of rigidity. Applications of the theory to improve pulp evaluation and to studies of important sheet properties (in the* **elastic** *regime) such as stiffness and sheet rigidity are discussed. A complex of phenomena of the* **plastic** *regime is inferred from theory. Stresses tending to cause rupture of fibre-to-fibre bonds are found in two important groups: those associated with torque on bonds resulting from shearing force in fibre segments (and combined with stress caused by anisotropic shrinkage of the fibres) and those associated with tension in fibres (and combined with the anisotropic shrinkage stress). The incidence of fibre-to-fibre bond rupture as the sheet strain is increased from the* **elastic** *range into the* **plastic** *regime is governed by equations developed for* **torque** *and* **tension** *bond failures. A brief discussion of the theory of the zero-span tensile test is included.*

Une étude théorique sur les propriétés mécaniques des structures fibreuses

On établit selon deux subdivisions principales—domaines élastique et plastique des relations effort-déformation—les théories *mathématiques de quelques propriétés mécaniques des structures fibreuses anisotropiques. Pour chacune des directions principales de la feuille les théories du régime élastique sont établies en fonction de la charge extérieure, du coefficient de Poisson, du module d'Young et du module de rigidité. Des applications de la théorie sont envisagées pour améliorer l'appréciation des pâtes et pour étudier les propriétés importantes de la feuille (sous régime élastique) telles que résistance à la flexion et la rigidité de la feuille. Un complexe de comportement du régime plastique est obtenu à partir de la théorie. Les forces tendant à provoquer la rupture des liaisons interfibres sont réparties en deux groupes importantes: celles qui sont associées à des couples sur les liaisons résultant des forces de cisaillement dans les segments des fibres (et ajoutées aux contraintes dues au retrait anisotropique des fibres) et celles qui sont associées aux tensions dans les fibres (et combinées également aux contraintes anisotropiques du retrait). Le nombre des ruptures de liaison qui se produit pendant une déformation croisante de la feuille depuis le domaine élastique jusqu'au domaine plastique, est gouverné par les équations établies pour la rupture des liaisons sous tension et torsion. Une brève discussion de la théorie de l'essai de traction 'zero-span' est inclue.*

Einige theoretische Betrachtungen über die mechanischen Eigenschaften von Faserstrukturen

Für die mechanischen Eigenschaften eines festen anisotropen Faservliesses wurden mathematische Theorien in zwei Hauptrichtungen, nämlich für den elastischen und den plastischen Bereich der Spannungs-Dehnungs-Beziehung entwickelt. Die Theorien für den elastischen Bereich wurden für jede der Hauptrichtungen in Blattebene für äussere Belastung, Poisson'sches Verhältnis, Young-Modul und Modul für Unbiegsamkeit aufgestellt und ihre Anwendung zur Verbesserung der Stoffcharakterisierung und zum Studium wichtiger Blatteigenschaften (im elastischen Bereich), wie Steifigkeit und Unbiegsamkeit, diskutiert. Von der Theorie wurde ein Komplex von Phänomenen des plastischen Bereiches abgeleitet. Man fand, dass diejenigen Spannungen, die zu einer Zerstörung der Zwischenfaserbindungen führen, zwei Hauptgruppen bilden, nämlich diejenigen, die mit einem auf die Bindungen wirkenden Drehmoment als Folge der in den Fasersegmenten auftretenden Scherkraft verbunden sind (und zwar zusammen mit der Spannung, die durch die anisotrope Schrumpfung

der Fasern hervorgerufen wird) und diejenigen, die mit der in den Fasern auftretenden Zugkraft verbunden sind (im Zusammenhang mit dem anisotropen, durch die Schrumpfung bewirkten Zug). Die Zerstörung von Zwischenfaserbindungen beim Übergang des Dehnungs Prozesses vom elastischen in den plastischen Bereich wird durch Gleichungen bestimmt, die für die Zerstörung der Bindungen durch Drehmoment und Spannung entwickelt wurden. Eine kurze Diskussion der Theorie der Null-Reisslängen-Prüfung schloss sich an.

Introduction

THE development of sound mathematical theories on the physical properties of paper and paperboard is obviously in the very interesting early phase that is characterised by high heuristic value and relatively low predictive power. In this stage, theories are fraught with suggestions for experimental observations; proper laboratory data, in turn, permit refinement of theoretical structures and, in later phases, it is usually found that the values of theory shift to embrace quantitative prediction of system properties and provide better measures of the properties of the structural components. It need hardly be said that the ultimate goals of theories on fibrous structures are improved end products and refined control of raw materials and process variables. We are now concerned, however, with the very early stage of development of theory and our attention must necessarily turn to rudimentary matters.

This paper presents the results of efforts towards improved theories on certain fundamental sheet properties. The composition of the paper is as follows: the first sections present theories appropriate to the *elastic* or Hookean stress/strain regime, yielding expressions for external load, Poisson's ratio, Young's modulus and the modulus of rigidity for the principal directions of an anisotropic sheet; the second set of sections presents discussions of the interfibre and intrafibre mechanisms with which one should deal to account for the *plastic* stress/strain regime; the paper concludes with a note on the theory of the zero-span tensile test.

Before turning to theories relating to the elastic regime, it might be helpful to observe that, while no theory can be expected to account perfectly for all the properties of anything as complex as a paper sheet (observations with a microscope are most discouraging!), the models for theories should have some verisimilitude to actual structures. In the author's opinion, models incorporating regular geometric arrays of filaments (such as a superposition of an orthogonal set aligned with the x and y axes and a second orthogonal set arranged at 45° with the axes) are ill-advised. An anisotropic distribution function for angular orientation and random spatial distribution for the fibres should be presumed, along with the most important distribution functions for such variables as fibre dimensions and interfibre bond spacing. The best efforts in the past (briefly reviewed in the next section) have incorporated such distribution functions. It should also be observed that the legitimacy of the assumptions and simplifying approximations underlying a theory should be judged from the point of view of the range of properties of the simulated product. An example from the following section is a theory for non-woven webs; it appears to account quite nicely for a number of the mechanical properties of non-woven fabrics and, no doubt, other low-density materials, but is believed by the author to be inadequate for papers of typical density. Similarly, it is expected that the author's simplifying assumptions will prove to be illegitimate when the interest centres on such high-density papers as greaseproof and glassine, tracing paper, etc.

The elastic or Hookean stress/strain regime

As is well known, the tensile stress/strain relationship for paper is very nearly linear for strains up to about 0·005 (0·5 per cent) when the duration of the test is not greater than a few minutes. We shall refer to the approximately linear, small-strain range as *elastic* and the range beyond about 0·005 as the *plastic* regime. Strictly speaking, the whole range involves creep and is therefore plastic; however, most load/elongation recordings for paper (particularly those for the cross direction) display two distinctly different regions and one finds the concepts of elastic and plastic regimes quite useful approximations to the true state of affairs.

A pioneering mathematical development was that of Cox.(1) One of his basic assumptions was that the fibres are attached only at their ends; thus, although his theory was useful for his purposes, it would be inadequate for ordinary papers. Onogi and Sasaguri,(2) one of whose interests was a proper accounting for paper thickness changes during straining, evolved a mathematical theory of a fibrous web. Their assumption to the effect that, during straining, fibre-to-fibre bonds undergo no angular displacement was found during the course of the present work to be unsuitable: in the special case of random distribution of fibre elements in the plane of the sheet, an important criterion based on a well-known relationship between Young's modulus, Poisson's ratio and the modulus of rigidity is not satisfied. More serious, however, is their tacit assumption, discernible at the outset of their theory, that the sum of the tensile and shear forces in a fibre element is a vector having direction parallel to that of the externally applied load.

A most interesting and thoroughgoing work is that of Petterson,(3, 4) whose interest was in developing theories to predict the mechanical properties of non-woven fabrics. In an important early phase of his work, he demonstrated that non-woven fabrics (commercial, anisotropic) obey Love's theory(5) for orthotropic materials. Of special interest is his mathematical treatment of a fibrous system in which fibre elements may (depending on orientation and direction of the external load) experience tension or compression. Although the bonding in the webs with which Petterson was concerned was not as extensive as that in ordinary papers, it was assumed that consolidation was sufficient to prevent buckling in fibre elements subjected to axial compression. The Petterson theory, probably adequate for both non-woven fabrics and papers of low density, has the weakness, when extended beyond its scope to typical paper, of omission of the effects of shear and flexure in fibre segments.

Model and assumptions for the present work

The theoretical treatments presented in the following sections are based on a model fibrous web that has been dried under restraint, so that the various forces in a fibre element come into existence with the initial infinitesimal straining of the whole web and are linear functions of the strain to which the web is subjected. The theory does not embrace a web that has been creped or dried without restraint (such as in loft drying) or mechanically treated to modify the configuration of the fibres in such manner that the foregoing assumption about the forces in fibres does not hold.

The fibre centres are distributed randomly over the area of the sheet, with an average number of N fibre centres lying in unit area. The fibres are assumed to be almost parallel to the surfaces of the sheet, with the fibre segments lying in a small angular range from such parallelism; *cosine error* associated with failure of this assumption is taken to be small. In this model, a *segment* is the interval along a fibre between points of bonding with other fibres. The angular orientation of fibre segments in the plane of the sheet is described by a distribution function P_θ (the chance that a fibre segment will lie between θ and $\theta+d\theta$ is $P_\theta d\theta$). In the present work, the principal directions of the anisotropic sheet are designated x (for example, machine-direction) and y (cross-direction) and P_θ is assumed to be symmetrical with respect to these directions. The z direction, about which little is done in the present work, is, of course, perpendicular to the xy plane. The angle θ is in the xy plane and is measured from the y direction (*see* Fig. 1).

The present treatment does not require that the whole fibre be straight. The analysis needs only the assumption that, over a fibre interval of one or two segments (a few per cent of the fibre length), the interval may be regarded as essentially straight. Beyond the obvious point that the analysis treats an *initially* straight segment, there are two considerations here of importance: the expression for the probability of involvement of a fibre in the analysis of the sheet stress is really the probability that a *segment*, of length between s and $(s+ds)$, will be 'cut' by a mathematical plane, rather than the probability that a whole fibre will be cut by a plane. The other consideration relates to the definition given above for P_θ; this function describes the distribution of angles of segments rather than of whole fibres. The most appropriate and undoubtedly the most objective experimental definition of P_θ lies in the technique of counting (with the aid of a microscope) short fibre intervals (of predetermined length, unrelated to segmental lengths).

The fibres of the model are assumed to be ribbon-like, with the broader dimension of the cross-section parallel to the xy plane. The fibres are not presumed to be of uniform size, but to have basic dimensions described by distribution functions. Following Onogi and Sasaguri,[2] variations in cross-sectional dimensions are taken into account by distribution functions in cross-sectional area (P_A) and moment of inertia of the fibre cross-section referred to a neutral axis in the z direction (P_I). The fibre segmental length s is described by a distribution function P_s. The structure and cross-sectional dimensions of a fibre are considered constant only within a segment; in assessing this from a statistical point of view it should be borne in mind that s is typically of the order of only a few per cent of the fibre length. However, the moduli of elasticity of the fibres, E for the axial direction (Young's modulus) and G, the modulus of rigidity for shear angle referred to the axis of the fibre and in the 'plane of the ribbon', are considered uniform throughout all the fibres.

As indicated earlier, it has been found necessary in the present work to provide for angular displacement of fibre-to-fibre bonds. It is assumed, however, that the linear displacement of one end of a fibre segment relative to the other end is simply that corresponding to the straining of the web. It is assumed, with earlier workers, that fibre segments subjected to axial compressive stress do not buckle. Ranger and Hopkins[6] hold the view that fibre-to-fibre bond failure may start with fibres under axial compressive stress and this implies buckling of a sort; this kind of action, however, is not to be expected at low levels of web straining. There is some evidence from laboratory work on stiffness (mostly unpublished), involving small strains, that might be explained by failure of this assumption to be strictly valid. The point is an important one and should be kept in mind in future refinements of theory.

Poisson's ratio ν_{xy} *and* ν_{yx} *and the principal moduli (Young's)* $\mathscr{E}_x$ and $\mathscr{E}_y$

A schematic representation of a fibre segment before (s) and after (s') straining when the web is subjected to a strain e in the x direction is given in Fig. 1. The initial angles of the fibres to which the fibre under consideration is

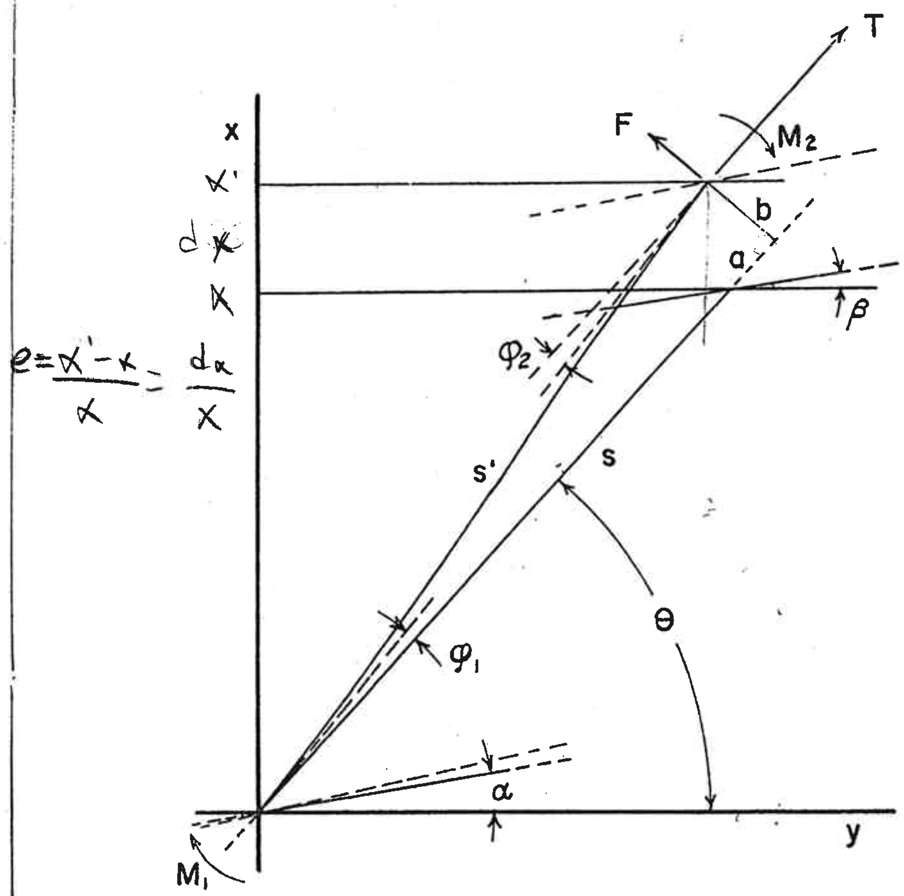

Fig. 1—Schematic representation of deformation of a fibre segment
The bonds undergo angular displacements ϕ_1 and ϕ_2, but are rigid so that $\theta-\alpha$ and $\theta-\beta$ remain constant

bonded at the 'lower' and 'upper' ends of the segment are α and β. From the geometry it is easily shown that—

$$a = es(\sin^2\theta-\nu_{xy}\cos^2\theta), \quad (1)$$

$$b = es(1+\nu_{xy})\sin\theta\cos\theta, \quad (2)$$

in which ν_{xy} is Poisson's ratio for the web, with reference to contraction in the y direction resulting from loading of the web in the x direction. On eliminating s from the right side of equation (1), one has a nearly exact expression (low-level straining of the web) for the axial strain in the fibre. When θ lies within $\pm \tan^{-1}\sqrt{\nu_{xy}}$ of the y axis the fibre is subjected to axial compressive stress; when the web is loaded in the y direction the axial compressive loading occurs in segments for which θ lies within $\pm\cot^{-1}\sqrt{\nu_{yx}}$ of the x axis. In all other segments the axial stress is tensile.

The displacement b is associated with flexure and shear in the segment. If the ends of the segment undergo angular displacements ϕ_1 and ϕ_2, it may be shown through analysis of the flexure and shear of the segment that—

$$b = Fs^3/12EI+(\phi_1+\phi_2)s/2+Fs/AG \quad . \quad . \quad . \quad . \quad (3)$$

where F is the shearing force (that is, the force acting over a fibre cross-section in a direction normal to the fibre axis as indicated in Fig. 1) and the other quantities have the earlier defined meanings. Equation (3) is really an approximation, but is very accurate for strains appreciably beyond those of the elastic regime, in so far as error associated with finiteness of strain is concerned. Of more significance is the error associated with the consideration that the segmental length s is not large in comparison with the fibre width (parallel to the xy plane); this is discussed by Timoshenko.(7)

The extension a is related to the tensile force T (compressive if negative) through equation (4)—

$$a = Ts/AE \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The angular displacements of the bonds at the two ends of the segment may be arrived at in the following manner. As straining of the web occurs, up to the onset of failure in a bond, it would appear that the angles between the segment and the 'lower' and 'upper' intersecting fibres should remain fixed at the initial values $(\theta-\alpha)$ and $(\theta-\beta)$, respectively. It is assumed that, on the average, the 'lower' fibre segment at the bond will undergo as much angular displacement at the bond as the lower end of the segment under consideration. A similar assumption is made for the bond at the 'upper' end of the segment. It may then be shown that—

$$\phi_1 = (e/2)(1+\nu_{xy})(\sin\theta\cos\theta+\sin\alpha\cos\alpha) \quad . \quad . \quad . \quad . \quad (5)$$

$$\phi_2 = (e/2)(1+\nu_{xy})(\sin\theta\cos\theta+\sin\beta\cos\beta) \quad . \quad . \quad . \quad . \quad (6)$$

Before considering the effect of varying α and β over all possible angles, it is of interest to note that there will be no bond rotation only when α (or β) is $(\pi-\theta)$; the bond rotation will approach its maximum possible value (for a

given θ) when the angles are equal and in that isolated case there is no flexure or shear in the segment. When the segment is along either of the two principal directions ($\theta=0$ or $\pi/2$), for which orientations there is nominally (and on the average) no flexure or shear in the segment, there may be appreciable flexure and shear when α and β are midway between 0 and $\pi/2$ or between $\pi/2$ and π.

Utilising equations (2, 5, 6) in equation (3), one obtains the following expression for the shearing force—

$$F = (e/2h)(1+\nu_{xy})[\sin\theta\cos\theta-(1/2)\sin\alpha\cos\alpha-(1/2)\sin\beta\cos\beta] \quad . \quad (7)$$

where

$$h = (s^2/12EI)+(1/AG) \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

To obtain the average value of F for all angles α and β, one may hold β constant and vary α—

$$\langle F\rangle_{Av} = (e/2h)(1+\nu_{xy})\,[\sin\theta\cos\theta-(1/2)\int_0^{\pi} P_\alpha\sin\alpha\cos\alpha\,d\alpha -(1/2)\sin\beta\cos\beta] \quad (9)$$

$$\beta = \text{const.}$$

where P_α is the angular distribution function. When β is varied over all angles, the expression obtained for the average shearing force involves a second integral [from the last term in equation (9)]. When the angular distribution function is symmetrical with respect to the principal axes (which is believed generally to be the case, at least as a good approximation), both integrals vanish; thus—

$$\langle F\rangle_{Av} = (e/2h)(1+\nu_{xy})\sin\theta\cos\theta \quad . \quad . \quad . \quad . \quad (10)$$

It is of interest to note that equation (10) yields exactly half of what one obtains on the assumption of no angular displacement of bonds.

Taking the components of the shearing and tensile forces along the x direction [calling into play equations (1), (4), (10)], one obtains for the contribution to the external load on the sheet—

$$\langle F\rangle_{Av}\cos\theta+T\sin\theta = (e/2h)(1+\nu_{xy})\sin\theta\cos^2\theta + eAE(\sin^3\theta-\nu_{xy}\sin\theta\cos^2\theta) \quad (11)$$

It is desirable at this point to introduce the number m of fibre segments per unit area (in the xy plane) of the sheet. The total length of fibre in unit area of the sheet in segments having length between s and $(s+ds)$ is readily seen to be $msP_s\,ds$; accordingly, if W is the mass per unit area of the sheet (basis weight in g/cm²), $\langle A\rangle_{Av}$ is the mean cross-sectional area of the fibres and ρ is the mean density of the fibre wall—

$$W = \langle A\rangle_{Av}\rho m \int_0^\infty sP_s\,ds \quad . \quad . \quad . \quad . \quad . \quad (12a)$$

$$m = W/(\langle A\rangle_{Av}\rho\langle s\rangle_{Av}) \quad . \quad . \quad . \quad . \quad . \quad (12b)$$

The sheet is now 'cut' with a yz plane (normal to the sheet stress). The number of segments having length between s and $(s+ds)$ cut by this plane is $\mathscr{U}msP_s \sin\theta\, ds$, where $\mathscr{U}$ is the width of the strip under tension; introducing this and the other distribution functions and integrating, one obtains for the total tensile force in the sheet—

$$\mathscr{T}_x = \frac{\mathscr{U}We_x}{\langle A\rangle_{Av}\rho\langle s\rangle_{Av}}\left[(1/2)(1+\nu_{xy})\int_0^\pi\int_0^\infty\int_0^\infty\int_0^\infty (s/h)\sin^2\theta \right.$$
$$\times\cos^2\theta\, P_\theta P_A P_l P_s\, ds\, dl\, dA\, d\theta$$
$$+\int_0^\pi\int_0^\infty\int_0^\infty AEs\sin^4\theta\, P_\theta P_A P_s\, ds\, dA\, d\theta$$
$$\left.-\nu_{yx}\int_0^\pi\int_0^\infty\int_0^\infty AEs\sin^2\theta\cos^2\theta\, P_\theta P_A P_s\, ds\, dA\, d\theta\right] \quad (13)$$

In equation (*13*), the strain e_x replaces the earlier e to avoid later confusion when strain is applied in the y direction; it should be borne in mind that h is a function of the variables s, l and A [equation (*8*)].

When the external load is applied in the y direction (the strip is cut in the y direction and the width $\mathscr{U}$ is now in the x direction), a treatment similar to that given above yields—

$$\mathscr{T}_y = \frac{\mathscr{U}We_y}{\langle A\rangle_{Av}\rho\langle s\rangle_{Av}}\left[(1/2)(1+\nu_{yx})\int_{-\pi/2}^{\pi/2}\int_0^\infty\int_0^\infty\int_0^\infty (s/h)\sin^2\theta \right.$$
$$\times\cos^2\theta\, P_\theta P_A P_l P_s\, ds\, dl\, dA\, d\theta$$
$$+\int_{-\pi/2}^{\pi/2}\int_0^\infty\int_0^\infty AEs\cos^4\theta\, P_\theta P_A P_s\, ds\, dA\, d\theta$$
$$\left.-\nu_{yx}\int_{-\pi/2}^{\pi/2}\int_0^\infty\int_0^\infty AEs\sin^2\theta\cos^2\theta\, P_\theta P_A P_s\, ds\, dA\, d\theta\right] \quad (14)$$

The y direction remains the reference for θ; for example, the direction of the external load is now along $\theta=0$.

The cumbersome integrals appearing in equations (*13*) and (*14*) will now be replaced with the following symbols—

$$\left.\begin{aligned}\mathscr{S}_{1x} &= \text{First integral, equation (13)}\\ \mathscr{S}_{2x} &= \text{Second integral, equation (13)}\\ \mathscr{S}_{3x} &= \text{Third integral, equation (13)}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (15)$$

$$\left.\begin{aligned}\mathscr{S}_{1y} &= \text{First integral, equation (14)}\\ \mathscr{S}_{2y} &= \text{Second integral, equation (14)}\\ \mathscr{S}_{3y} &= \text{Third integral, equation (14)}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (16)$$

Expressions for Poisson's ratio for the two principal directions may now be obtained. First, with the sheet under load in the x direction, the contribution of a fibre segment in the y direction is considered—

$$-\langle F\rangle_{Av}\sin\theta+T\cos\theta = -(e_x/2h)(1+\nu_{xy})\sin^2\theta\cos\theta$$
$$+e_xAE(\sin^2\theta\cos\theta-\nu_{xy}\cos^3\theta) \quad (17)$$

It is seen that the number of segments cut by an xz plane in unit distance along the x direction is $msP_s\cos\theta\, ds$. Introducing this and the other distribution functions and integrating, one obtains an expression for the total contribution in the y direction. This is equated to zero; after cancelling a factor from the equation—

$$(1/2)(1+\nu_{xy})\mathscr{S}_{1y}+\nu_{xy}\mathscr{S}_{2y}-\mathscr{S}_{3y} = 0 \quad . \quad . \quad . \quad . \quad (18)$$

In similar manner, through consideration of the total contribution of force in the x direction when the sheet is loaded in the y direction, one finds—

$$(1/2)(1+\nu_{yx})\mathscr{S}_{1x}+\nu_{yx}\mathscr{S}_{2x}-\mathscr{S}_{3x} = 0 \quad . \quad . \quad . \quad . \quad (19)$$

Equations (*18*) and (*19*) yield expressions for Poisson's ratio for the principal directions—

$$\nu_{xy} = \frac{2\mathscr{S}_{3y}-\mathscr{S}_{1y}}{\mathscr{S}_{1y}+2\mathscr{S}_{2y}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$\nu_{yx} = \frac{2\mathscr{S}_{3x}-\mathscr{S}_{1x}}{\mathscr{S}_{1x}+2\mathscr{S}_{2x}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Equations for Young's modulus are obtainable immediately from equations (*13*) and (*14*)—

$$\mathscr{E}_x = \frac{W}{\mathscr{Z}\langle A\rangle_{Av}\rho\langle s\rangle_{Av}}\left[\frac{(1+\nu_{xy})}{2}\mathscr{S}_{1x}+\mathscr{S}_{2x}-\nu_{xy}\mathscr{S}_{3x}\right] \quad . \quad . \quad (22)$$

$$\mathscr{E}_y = \frac{W}{\mathscr{Z}\langle A\rangle_{Av}\rho\langle s\rangle_{Av}}\left[\frac{(1+\nu_{yx})}{2}\mathscr{S}_{1y}+\mathscr{S}_{2y}-\nu_{yx}\mathscr{S}_{3y}\right] \quad . \quad . \quad (23)$$

In these expressions $\mathscr{Z}$ is the sheet thickness.

When, as is generally the case, P_θ is symmetrical about the principal axes, it can be shown that

$$\left.\begin{aligned}\mathscr{S}_{1x} &= \mathscr{S}_{1y}\\ \mathscr{S}_{3x} &= \mathscr{S}_{3y}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

However, $\mathscr{S}_{2x}$ and $\mathscr{S}_{2y}$ may be substantially different, the former being the larger when, as is ordinarily the case in machine-made paper and board, P_θ is maximum along the x axis and minimum along the y axis.[8] When the sheet (for example, an ordinary handsheet) is isotropic in the xy plane, $P_\theta = \text{const.} = 1/\pi$ and these two integrals are equal.

The familiar reciprocity relation—

$$\nu_{xy}/\mathscr{E}_x = \nu_{yx}/\mathscr{E}_y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

may easily be shown to be satisfied by the foregoing theory. Satisfaction of this equation is a necessary but not sufficient condition that a theory be correct.

The modulus of rigidity $\mathscr{G}_{xy}$ $(= \mathscr{G}_{yx})$

There is an increasing need, especially in engineering work involving paperboard and structural board, for theoretical and experimental information on the modulus of rigidity (for shear strain in the xy plane*).

Two kinds of diagram of distortion of the web corresponding to shear strain in the xy plane immediately come to mind. In the first and perhaps more conventional diagram, one visualises small displacements that are parallel to, say, the y axis, with lines initially parallel to the x axis rotating through a small angle γ (the shear strain). If one assumes that bonds undergo no angular displacement (a more obviously questionable assumption where the angular displacements of shear are concerned), the theory leads in an apparently straightforward manner to an expression for $\mathscr{G}_{xy}$. However, the theory does not satisfy the elementary criterion for the isotropic case that $\mathscr{E} = 2(1+\nu)\mathscr{G}$. When angular displacement of bonds is introduced, the theory satisfies the foregoing criterion, but some uncertainty arises as to whether or not angular displacements are properly handled. The second diagram of distortion avoids this uncertainty through reference of angles to the principal axes of shear as shown in Fig. 2; these axes are designated 1–1′ and 2–2′. As is well known, the principal axes remain perpendicular to each other during shear; tension stress and strain ($\gamma/2$) exist along one axis (1–1′ in Fig. 2) and a numerically equal compressive stress and strain ($-\gamma/2$) exist along the other axis (2–2′ in Fig. 2). Only the second treatment is given here (it yields the same final result as the first).

* There should be utility, too, in theory on $\mathscr{G}_{xz}$ and $\mathscr{G}_{yz}$; no attempt was made in the present work to develop theoretical expressions for these moduli.

As shown in Fig. 2, a fibre segment of length s is extended (literally shortened if δ lies between $\pi/4$ and $3\pi/4$) and rotated. The ends of the segment rotate through angles ϕ_1 and ϕ_2, the values of which depend on δ and the orientations α' and β' of the neighbouring segments with respect to axis 1–1′. Analysis of the diagram shows that the axial extension of the fibre is—

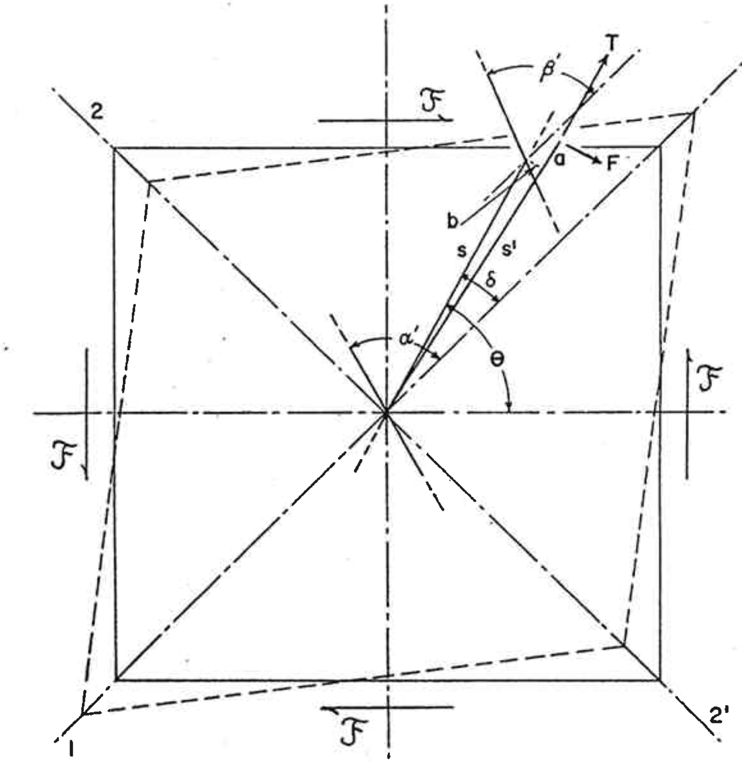

Fig. 2—Schematic representation of deformation of a fibre segment during shear in the xy plane

Although the bonds are rigid, they undergo angular displacement (ϕ_1 and ϕ_2 are not shown in this sketch)

$$a = (s\gamma/2)(\cos^2\delta - \sin^2\delta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$
$$(= s\gamma \sin\theta\cos\theta)$$

and the relative displacement (at the 'upper' end of the segment) normal to the axis is—

$$b = s\gamma \sin\delta\cos\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Assuming, as in the foregoing section, that the neighbouring segments share in the flexing with the segment under consideration, it may be shown that—

$$\phi_1 = (\gamma/2)\,(\sin\delta\cos\delta + \sin\alpha'\cos\alpha') \quad . \quad . \quad . \quad . \quad (28)$$

$$\phi_2 = (\gamma/2)\,(\sin\delta\cos\delta + \sin\beta'\cos\beta') \quad . \quad . \quad . \quad . \quad (29)$$

On substituting these angles in equation (3), utilising equation (27) and referring angles to the y axis ($\delta=\theta-\pi/4$; $\alpha'=\alpha-\pi/4$; $\beta'=\beta-\pi/4$), one obtains for the shearing force—

$$F = (\gamma/2h)\,[\sin^2\theta-(1/2)\sin^2\alpha-(1/2)\sin^2\beta].$$

Averaging over all values of α and β—

$$\langle F\rangle_{\mathrm{Av}} = (\gamma/2h)\left[\sin^2\theta-(1/2)\int_0^\pi P_\alpha\sin^2\alpha\,d\alpha-(1/2)\int_0^\pi P_\beta\sin^2\beta\,d\beta\right] \quad (30)$$

Of course, the two integrals in equation (30) are equal and, hence, the expression for the mean shearing force may be simplified to—

$$\langle F\rangle_{\mathrm{Av}} = (\gamma/2h)\left[\sin^2\theta-\int_0^\pi P_\alpha\sin^2\alpha\,d\alpha\right] \quad . \quad . \quad (30a)$$

The contribution of the shearing and tensile forces to the shearing force in the sheet is—

$$\langle F\rangle_{\mathrm{Av}}\sin\theta+T\cos\theta=(\gamma/2h)\,(\sin^3\theta-\sin\theta\int_0^\pi P_\alpha\sin^2\alpha\,d\alpha)$$
$$+\gamma AE\sin\theta\cos^2\theta \quad (31)$$

The last term in equation (31) invokes the axial strain in the segment a/s from equation (26). Utilising equation (12b) in the expression $msP_s\sin\theta\,ds$ for the number of segments cut by a yz plane of unit length in the y direction, introducing the distribution functions, integrating and dividing the total shearing force in the sheet by the strain and area, it is found that—

$$\mathscr{G}_{xy} = \frac{W}{\mathscr{Z}\langle A\rangle_{\mathrm{Av}}\rho\langle s\rangle_{\mathrm{Av}}}\Bigg[\int_0^\pi\int_0^\infty\int_0^\infty\int_0^\infty (s/2h)\sin^4\theta\,P_\theta P_A P_I P_s\,ds\,dI\,dA\,d\theta$$
$$-\int_0^\pi P_\alpha\sin^2\alpha\,d\alpha\int_0^\pi\int_0^\infty\int_0^\infty\int_0^\infty (s/2h)\sin^2\theta\,P_\theta P_A P_I P_s\,ds\,dI\,dA\,d\theta$$
$$+\int_0^\pi\int_0^\infty\int_0^\infty AEs\sin^2\theta\cos^2\theta\,P_\theta P_A P_s\,ds\,dA\,d\theta\Bigg] \quad . \quad . \quad . \quad . \quad (32)$$

Two necessary (although insufficient) conditions for the correctness of the theory are (a) $\mathscr{G}_{xy}=\mathscr{G}_{yx}$ and (b) in the isotropic case, for which $P_\theta=$ const. $=1/\pi$, $\mathscr{E}=2(1+\nu)\mathscr{G}$. In a test of the first condition, an expression was developed for $\mathscr{G}_{yx}$; the result is easily expressed by stating that the sines in all the integrals of equation (32), excepting the last, are changed to cosines and the limits of integration for θ and α are changed from $0\to\pi$ to $-\pi/2\to\pi/2$. It can then be shown that $\mathscr{G}_{xy}=\mathscr{G}_{yx}$, providing that P_θ is symmetrical with respect to the x and y directions. The second condition can be tested by putting $P_\theta=1/\pi$ in all the integrals of equation (15) or (16) and equation (32); although the mathematical manipulations become tedious, it can be shown that the second condition is satisfied.

Creep

When the duration of recording of the load/elongation characteristic of paper is substantially longer than a few minutes, creep is significant. As in other high polymers, creep in paper exists at all levels of stress and, when the strain is less than about 0·005, there seems to be no doubt that the creep is almost purely an intrafibre phenomenon. The following, therefore, is suggested for studies of creep phenomena in fibres or, alternatively, for predictive purposes when basic creep data are available. The moduli of elasticity given by equations (22), (23) and (32) are regarded as so-called *delayed moduli*—that is, in each case the delayed modulus is the quotient of the applied stress and the strain after a known duration t of the stress. To be sure, this implies an accurately linear relationship between the delayed strain (at some fixed time, t) and the stress and this will not generally exist;[9] however, it would be of interest to accept the approximation for at least the earlier phases of studies on the relationships between creep phenomena in fibres and sheets.

Some computational and numerical aspects

The theory developed so far brings to light the potentially important properties of fibres and sheet structure. As always supposed and as demonstrated by the recent work of Petterson,[3, 4] Young's modulus E and the cross-sectional area A of the fibres are very important. In papers having density in the ordinary range (but not in non-woven fabrics of the sort with which Petterson worked), the moment of inertia I of the fibre section (referred to an axis in the z direction) plays a significant rôle. Somewhat surprising is the influence of the rigidity modulus G of the fibre for shear strain in an xy plane: in the two terms comprising h, the second (involving G) is of the same order as the first (involving EI). This suggests that an array of fibre properties to be studied should include G; as direct measurement may prove to be quite difficult, evaluation through the theory (working through $\mathscr{E}_x$ and $\mathscr{E}_y$ or $\mathscr{G}_{xy}$ or all three moduli) might be attempted.

Fibre length does not appear in the equations developed for the *elastic* regime. According to the theory, the fibre length and the fibre length distribution function are not important, providing that the fibre length is large in comparison with the mean segmental length $\langle s \rangle_{Av}$. This condition is met in ordinary papers.

Until experimental and theoretical information on P_s is obtained, one must rest content with the use of the mean segmental length. In all the expressions for the moduli of elasticity and Poisson's ratio, $\langle s \rangle_{Av}$ factors out and cancels, leaving this quantity only in the first term of h; needless to say, P_s is dropped from the integrands when the mean segmental length is used. A mathematical treatment relating bonded area in a sheet with the mean segmental length is given in Appendix 2. The situation, however, is different with the remaining distribution functions. There are techniques available for determining the function P_θ[8] and for measuring the dimensions of cross-sections of fibres.* From the latter, the functions P_I and P_A can be calculated. In careful researches involving the theory, one should include these functions in the integrands (where indicated) and numerical integrations should be performed.

The theory has not accounted for the fact that fibres lying 'in the surfaces' of the sheet (practically none would lie *on* a surface with no external crossings of other fibres) have fewer points of bonding with other fibres. The number of points of bonding for surface fibres would seem to lie between the limits 0·5 to a fraction approaching unity of the number of bonding points for internal fibres. It would seem that at least an approximate correction to the theory (important in thin sheets) could be worked out. A consequence of this consideration [realised on studying equations (*13*) and (*14*)] is that Young's modulus of an imagined thin layer in an anisotropic sheet is at a maximum when the layer is at the midplane and remains constant as the layer is moved towards either surface until 'surface fibres' enter the layer, when the modulus diminishes to a minimum value at the surface.

A numerical illustration

It is emphasised that the primary purposes of the theory at the present time relate to research. As suggested above, the several distribution functions should be taken into account in careful work. In order that the reader may gain some numerical feeling about the theoretical expressions, however, some calculations have been made on the basis of the following model: an isotropic sheet comprised of uniform, ribbon-like fibres, all having width 0·0030 cm and thickness 0·0008 cm. From the work of Jayne,[11] a typical value of E for woodpulp fibres is seen to be about 3×10^{11} dyn/cm². As a pure guess, the modulus of rigidity of woodpulp fibres (for shear strain in the xy plane) is taken to be 1×10^{11} dyn/cm². Since an isotropic sheet was assumed for the purposes of calculation, $P_\theta=1/\pi$. The quotient $W/\mathscr{L}$ appearing in equations (*22*) and (*23*) for Young's modulus in the two principal directions and in equation (*32*) for the modulus of rigidity is, of course, the sheet density. This is taken to be 0·75 g/cm³ and the density of the fibre is assumed to be 1·5 g/cm³. Since the calculations are based on average quantities, $\langle s \rangle_{Av}$ appears only in the first term of h; this mean is taken equal to 0·005 cm (arrived at through the theory given in Appendix 2). The results of the calculation are—

$$\nu_{xy} = \nu_{yx} = \nu = 0{\cdot}298$$

$$\left.\begin{aligned} \mathscr{E}_x &= \mathscr{E}_y = 5{\cdot}27\times10^{10}\ \text{dyn/cm}^2 \\ \mathscr{G}_{xy} &= \mathscr{G} = 2{\cdot}03\times10^{10}\ \text{dyn/cm}^2 \end{aligned}\right\} \quad . \quad . \quad . \quad (33)$$

There are no experimental data available for a legitimate comparison with theory. In his recent dissertation work, Schulz[12] obtained Young's modulus for handsheets that had been dried under constant strain. The optimum 'degree of wet straining' for a handsheet of density comparable with that of the example yielded a Young's modulus of $5{\cdot}5\times10^{10}$ dyn/cm² (after correcting for the fact that Schulz based his stress calculations on the cross-sectional area of cellulose of density 1·55 g/cm³). The agreement is fortuitously good in view of the fact that the fibre properties and dimensions employed in the calculations were merely typical and were not based on the properties of the pulp employed in Schulz's work.

Applications of the theory for the elastic regime

The value of theory in research on the relationships between fibre and sheet properties has already been discussed. For some time to come, it is expected that the chief use of physical theories of paper will be in bringing to light hitherto neglected fibre characteristics and behaviour and in yielding improved techniques for measuring the visco-elastic properties of fibres; it is hoped and expected that theory will indicate improved techniques for pulp evaluation.

Working through theory in the other direction, towards the prediction of sheet properties, it should be brought to mind that a number of important behavioural properties of paper and paperboard involve stresses and strains in the *elastic regime*. Stiffness or flexural rigidity is one such property. In one's

* Dyson[10] has recently published the principles of a new optical device (referred to by the manufacturer as the Dyson image-splitting eyepiece), which, it is expected, would improve the accuracy of fibre mensuration by an order of magnitude.

thinking on this subject, it should be borne in mind that equations (*22*) and (*23*) should be applicable to mathematical laminae within a sheet; when recognition is made of the fact that density and all the distribution functions change throughout the thickness of a sheet, this point of view is theoretically preferred—especially in work on stiffness. In using equations (*22*) and (*23*) in this way, the factor $W/\mathscr{Z}$ should be replaced with ρ_{sheet}, the density of the web in a given plane (treated as a variable throughout the sheet). (In the purest application of theory, whether one is concerned with stiffness, Young's modulus in pure tension or the modulus of rigidity, this point of view should be used.)

Other properties of importance in the *elastic* regime are the rigidity of paperboard and structural fibre board for shear in the *xy* plane; this is of growing engineering importance in theories relating to containers and building materials.

In many converting operations in which a rapidly moving web is quickly subjected to strain, the level of strain is in the *elastic* regime and, because of the high rate of straining, there is reason to hope that theory will provide a reliable basis for developmental work.

The plastic stress/strain regime

USUALLY, but not always, the stress/strain characteristic of paper displays a second, nearly linear portion whose slope is much less than that of the *elastic* regime.[13] In the case of non-woven webs comprised of synthetic fibres, Petterson has shown that the plastic regime can be associated with the plastic nature of the individual fibres.[3, 4] It is not generally believed that the plastic regime of paper and paperboard made from wood fibre can be attributed to the plastic nature of the individual fibres; the stress/strain curves for individual fibres display much less curvature than that of typical paper. There have been opposing schools of thought on the mechanisms operating in a paper sheet under stress that may account for the stress/strain characteristic and for creep and creep-recovery data. It is held by some that both the short-term and long-term strain exhibited by paper at low levels of stress is attributable almost entirely to elastic and creep deformation of the fibres themselves. Others feel that the fibres may be regarded as essentially inextensible and that creep and plastic flow, particularly at high stress levels, is the result of fibre-to-fibre bond breaking and fibre-to-fibre slipping. The former view is very well substantiated (for low levels of stress and strain) by the excellent data on the stress/strain relationship and creep obtained by Steenberg and his associates at the Swedish Forest Products Research Laboratory[14–17] and by Brezinski at The Institute of Paper Chemistry.[9] The imaginative work of Nordman and his associates at the Finnish Pulp and Paper Research Institute[18–21] has furnished strong evidence that, at higher levels of stress and strain, the response of paper in the plastic region is attributable to breaking of fibre-to-fibre bonds. Very recently, using an ingenious microscopic technique adapted from a method developed by Emerton and Watts,[22] Page [23] and Page and Tydeman[24] at the British Paper and Board Industry Research Association have obtained information supporting the evidence that fibre-to-fibre bonds fail in the plastic regime. In connection with the pulp evaluation programme at The Institute of Paper Chemistry, Lathrop[25] has developed very sensitive photoelectric equipment to study the change in light flux scattered by single fibres as a function of straining. He observed both positive and negative changes on loading fibres and concluded that the probable net effect is too small to account for the observations of Nordman *et al.* Thus, it is concluded that the major portion of the change in light-scattering coefficient of paper produced by straining is attributable to fibre-to-fibre bond breaking. For the purposes of the present work, it is assumed that fibre-to-fibre bonds (in all but the weakest, lowest density papers) hold without failing for strains up to and perhaps beyond about 0·005; as treated in the theoretical sections for the *elastic* regime, each fibre-to-fibre bond is rigid, but undergoes slight angular displacement as the sheet is strained and the fibres are subjected to the actions of extension, flexure and shear; it is presumed, in view of the experimental evidence, that the onset of the plastic regime is occasioned by the breaking of an appreciable number of fibre-to-fibre bonds. Following Nordman and others, it is presumed that each increment of strain is associated with an increment in the number of bonds broken; intrafibre creep continues to contribute to the sheet strain; as the process of straining is continued to higher strains, a further mechanism—fibre failure—is considered possible; ultimately, of course, the sheet is so weakened by general bond failure and fibre rupture that disruption of the sheet commences at some point and quickly propagates across the strip.

A theory of the plastic regime should rest upon analyses of the stresses in fibre-to-fibre bonds arising from the several kinds of forces existing between fibres. The author is of the opinion that energy considerations alone are insufficient for a suitable mathematical treatment of the plastic regime. The mechanical interactions between fibres considered in the present work are the following—

1. Stresses in bonds owing to flexure of fibre segments.

2. Stresses in bonds resulting from the anisotropic shrinkage of fibre segments.

3. Stresses in bonds owing to tension in fibres.

The stress distribution in a bond is reckoned, of course, from a suitable summing of the foregoing stresses.

Stresses in bonds owing to flexure of fibre segments

According to the theory presented below, it develops that a very important aspect of flexure in fibre segments is the high shearing stress existing in certain bonds in consequence of the associated torque acting on the bonds. An estimate of the stress can be made in the following way. The contributions to the torque in the 'lower' bond shown in Fig. 1 from both the segment shown and the segment beyond the bond (assumed for illustrative numerical purposes also to have length s) is, from considerations similar to those leading to equation (*7*)—

$$Fs/2+F's/2 = (es/2h)(1+\nu_{xy})[\sin\theta\cos\theta-(1/2)\sin\alpha\cos\alpha - (1/4)\sin\epsilon\cos\epsilon-(1/4)\sin\beta\cos\beta] \quad (34)$$

in which F' is the shearing force in the extension of the fibre into the next segment beyond and 'below' the bond and ϵ is the orientation of the next crossing fibre beyond the bond. Numerical considerations of equation (*34*) show that the bond torque can vary from zero (when all three angles are equal) to a maximum value when $\theta=45°$ and $\alpha=\beta=\epsilon=135°$ (or, of course, when the values are interchanged). Using the numerical values leading to equations (*33*), it is found that this torque is about 3·5 dyn cm when the strain in the sheet is $e=0{\cdot}01$. The calculation of the maximum shear stress in a bond between two ribbon-like fibres corresponding to this torque—if done with rigour, taking into account the deformability of the fibre wall material —would in itself be a major effort. An approximate idea of the stress can be gained by treating the fibres as rigid and the bonded zone as disc-like. If τ_m is the maximum shear stress and if the stress at distance r from the centre of the zone is assumed to be proportional to r, it may be shown that—

$$\tau_m = (F+F')s/\pi R^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

in which the numerator is obtainable from equation (*34*). When the torque is 3·5 dyn cm and $R=0{\cdot}0015$ cm, the peripheral shear stress is found to be $6{\cdot}6\times10^8$ dyn/cm² (9 600 lb/in²). If the bond is held under this stress, its strength would be greater than that of the strongest adhesives determined with *macroscopic* areas.[26] When the stress discussed in the following section is taken into consideration, it will be seen that the maximum peripheral stress should be appreciably greater than this value.

Stresses in bonds resulting from anisotropic shrinkage

It is well known that woodpulp fibres shrink substantially in transverse directions (on drying from equilibrium with water) while displaying only a small shrinkage in the axial direction; inspection of the data of Weidner,[27] for example, shows that a typical transverse contraction of about 5 per cent

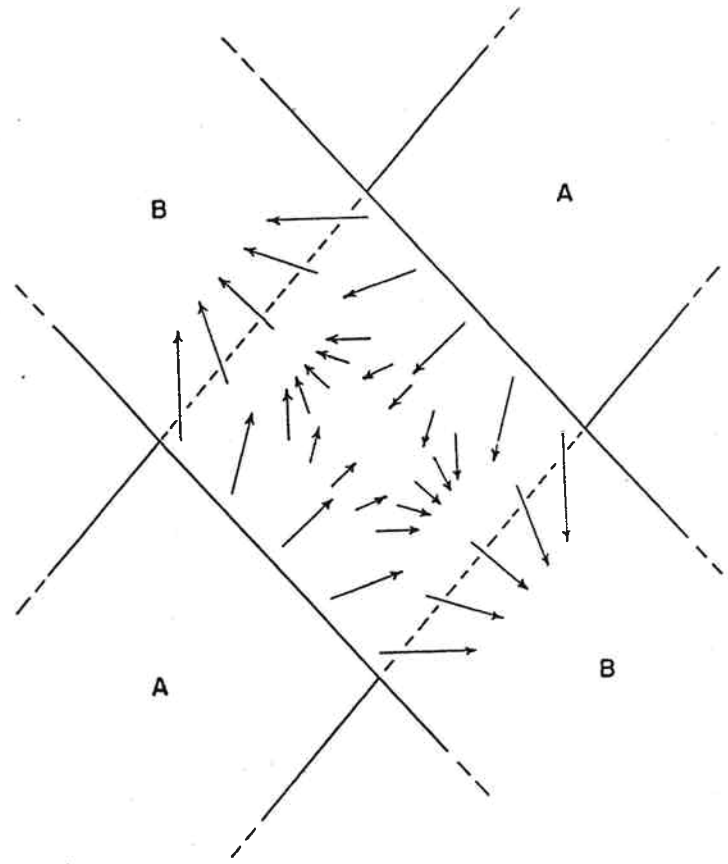

Fig. 3—Shearing forces in a bond, shown for only one fibre *A*, caused by anisotropic shrinkage: a similar set, with signs reversed, should exist for fibre *B*

occurs for woodpulp fibres on drying from water to equilibrium with air at 50 per cent relative humidity. It is not known at what moisture content the Campbell effect[28] has completed its task of bringing molecular bonds into action, but it is presumed that, at this point, only a fraction of the ultimate shrinkage of the fibre has occurred. At a bond like that pictured in Fig. 3 the differential shrinkage will cause a distribution of shearing forces (to be visualised as acting on elemental areas) on each fibre like that shown for fibre *A*. In view of considerations similar to those made by deBruyne[26] in his interesting discussion of stress-concentrating effects in glued joints, one must conclude that the stresses established in fibre-to-fibre bonds by drying of the sheet are maximum at the periphery.

The order of magnitude of the maximum stress arising in anisotropic shrinkage can only be estimated. Involved in the estimate are the axial and transverse moduli of elasticity of the fibres, the effective strain in each fibre and the geometry. On the basis of the structure of a fibre, it is presumed that the axial value of Young's modulus is much greater than the transverse. Thus, for an order-of-magnitude estimate, it may be assumed that the axial compression of either fibre is small compared with the effective transverse extension of the other. Let it be supposed that the latter is 0·03 and put E_t at only $0{\cdot}5 \times 10^{11}$ dyn/cm², half the fibre thickness at 0·0004 cm and the fibre width at 0·003 cm.

The computed *average* shearing stress, from one centre line along the other to the periphery, is seen to be 4×10^8 dyn/cm² (5 800 lb/in²). The stress at the periphery of the bond should be higher than the average value; however, creep and stress relaxation will lessen the stress and, hence, we might think of the calculated stress given above as a reasonable estimate of the order of magnitude of the stress caused by differential shrinkage of the fibres. The stress is at a surprisingly high level.

Combination of bond stresses originating in torque and anisotropic shrinkage

The vector sums of the stress forces shown in Fig. 3 and those arising in torque are qualitatively shown in Fig. 4. It is of considerable interest to note that the peripheral shearing stress displays two maxima and two minima, estimated to be about 10^9 dyn/cm² (15 000 lb/in²)—for a sheet strain of 0·01. In judging the possible effect of a shearing stress at this high level, it should be borne in mind that rupture stresses for very small areas tend to be appreciably higher than are observed for ordinary, macroscopic areas: it may well be that a fibre-to-fibre bond measuring only 30 μ on a side could withstand a shearing stress as large as the foregoing, but this seems doubtful.

It seems clear that bond failure resulting from shearing stress due to fibre flexure and anisotropic shrinkage should generally initiate at the ends of the diagonal that is the more closely perpendicular to the external tensile load (Fig. 4).

As the straining of the sheet increases beyond the *elastic* regime, some of the bonds between fibres that are roughly perpendicular to each other and oriented at approximately 45° and 135° will fail and, as will be seen later, it is

expected that some of the bonds near the ends of fibres that are roughly parallel to the external load will fail. As the straining is increased, *torque failures* will occur for fibres oriented farther and farther away from the angular orientations yielding maximum torque, in accordance with equation *(34)*

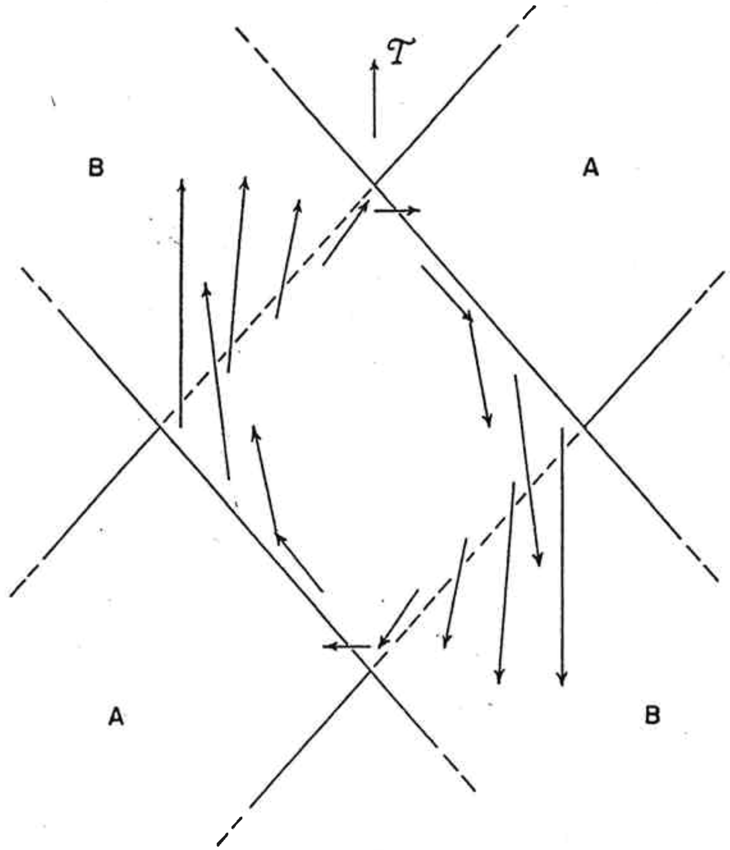

Fig. 4—Shearing forces in a bond, shown for only one fibre *A*, resulting from the combined effects of torque and anisotropic shrinkage

It is expected that torque bond failures should initiate in the left or right limbs, where the shearing stress is maximum (only the peripheral forces are shown)

and *tension failures* will occur for fibres that are oriented farther away from parallelism with the external load. It would seem that torque failures may occur with nearly equal probability anywhere along a fibre, whereas (as will be shown in the next section) a tension failure of a bond will most probably occur at either end of a fibre.

Before proceeding to the discussion of tension failures of bonds, a numerical aspect of equation (*34*) should be given attention. When $\sin\theta\cos\theta$ is at a maximum value (either in the first or second quadrant) and the other products are of opposite algebraic sign, but numerically at their maxima, the angles can deviate appreciably from 45° or 135° without significant change in these products. For example, the angle may change from 32° to 58° or from 122° to 148° with a torque reduction of only 10 per cent (the range is 18° corresponding to a variation of only 5 per cent). It would seem, then, that the first torque failures would involve statistical fluctuations in the strength of bonds for fibre segments occurring in appreciable ranges of orientation angle. At greater strains, according to equation (*34*), a wide variety of angular orientations of bonded segments could result in torque failures of bonds.

Stresses in bonds owing to tension in fibres

A consideration of the equilibrium of a fibre leads immediately to the conclusion that the tension (or compression) in a fibre at any point is the algebraic sum of the shear forces in the bonds from that point to either end of the fibre. It is important to consider the manner of build-up of fibre tension, because the shear force at a bond is equal to the increment in fibre tension from one side of the bond to the other. If the build-up is gradual, the shear force is weak; if it is sudden, the shear force is strong. Of interest are the mathematical aspects of the special, idealised case pictured in Fig. 5. All the segments shown are of the same length s and of similar cross-sectional dimensions. Each crossing fibre is treated as a beam of length $2s$ having clamped ends and these ends are assumed to move with the sheet extension—that is, after the sheet has been subjected to a strain e, neighbouring ends of adjacent 'beams' move apart from an initial separation s to $s(1+e)$.

Numbering from the 'top' of the fibre, there are n segments to the middle of the fibre in which to reckon the build-up of tension. In the first segment, the shear force $(\delta T)_1$ is also the tension; in the second, $T_2=(\delta T)_1+(\delta T)_2$, etc. The strain in the pth segment is given by equation (*36*) and the following equations relate the shear forces (the δT values) and the tensions. In equation (*36*), x_p is the central deflection of the pth crossing fibre (Fig. 5).

$$e_p = e-(x_p-x_{p+1})/s \quad \ldots\ldots \quad (36)$$

$$\left.\begin{aligned}(\delta T)_1 &= cx_1\\ (\delta T)_2 &= cx_2\\ \vdots\quad & \quad\vdots\\ (\delta T)_p &= cx_p\end{aligned}\right\} \quad \ldots\ldots \quad (37)$$

$$\left.\begin{aligned}T_1 &= \quad(\delta T)_1 &&= c'e_1\\ T_2 &= T_1+(\delta T)_2 &&= c'e_2\\ \vdots\quad & \qquad\vdots && \quad\vdots\\ T_p &= T_{p-1}+(\delta T)_p &&= c'e_p\end{aligned}\right\} \quad \ldots\ldots \quad (38)$$

$$(\delta T)_p = sce+(\delta T)_{p+1}-(sc/c')T_p \quad \ldots\ldots \quad (39)$$

Equation (*39*) is derived from the three preceding sets of relationships. Now,

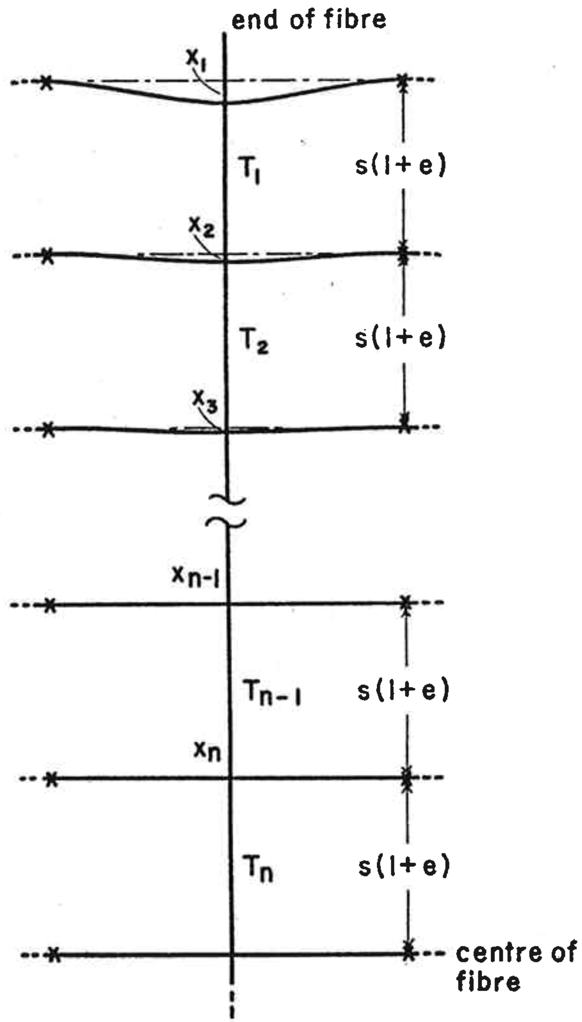

Fig. 5—Disposition of a fibre and crossing fibres (most flexible arrangement) appropriate to the development of equations (*36–42*) at the centre of the fibre, $x_{n+1}=0$, by symmetry and $(\delta T)_{n+1}=0$. Then, for the nth segment (putting $sc/c'=j$)—

$$(\delta T)_n = sce - jT_n.$$

Alternating from $T_{n-1}=T_n-(\delta T)_n$ [that is, from equation (38)] to equation (39) to obtain $(\delta T)_{n-1}$, then to equation (38) to obtain T_{n-2}, etc., one can obtain a series of expressions for the tensions in the segments and the shear forces in the bonds. Before illustrating this, let us put $T_n=\eta c'e$, in which $c'e$ is the maximum possible tension in a fibre aligned in the direction of the sheet strain (corresponding to an infinite number of points of bonding) and η is the fraction of this maximum possible tension existing at the centre of the fibre. [If one wishes to consider a fibre at angle θ, one substitutes for 'e' the expression for a/s from equation (1)]. For the fifth segment from the fibre centre, for example, one finds—

$$T_{n-4} = [\eta-(1-\eta)(10j+15j^2+7j^3+j^4)]c'e \quad . \quad . \quad . \quad (40)$$

$$(\delta T)_{n-4} = (1-\eta)(5j+20j^2+21j^3+8j^4+j^5)c'e \quad . \quad . \quad . \quad (41)$$

The scheme for obtaining the numerical factors of j and its powers is given in Fig. 6. This has been worked out for values of n up to 10; it is easily extended to larger values. The number q is defined as $(n-p)$, so that $q=0$ for the segment nearest the fibre centre (the nth) and $q=9$ for the segment nearest the fibre end in an example with $n=10$ (whole length of fibre bonded at 21 points).

The value of η can be obtained by developing the expressions for T_1 and $(\delta T)_1$; these are equated and η is computed. It then becomes possible to compute other segmental tensions and shear forces.

From equations (37 and 38), which define c and c', it is readily seen that—

$$c = 24EI/s^3 = 1{\cdot}04\times10^8 \text{ dyn/cm},$$

$$c' = AE = 7{\cdot}2\times10^5 \text{ dyn},$$

when the fibre cross-section and Young's modulus employed earlier for illustrative purposes are used and $s=0{\cdot}005$ cm. It is then seen that $j=sc/c'=0{\cdot}72$. When the expressions for T_1 and $(\delta T)_1$ are equated, with $n=10$ and $j=0{\cdot}72$, computation shows that—

$$\eta = 1-\frac{1}{2661} = 0{\cdot}999625.$$

In other terms, the tension built up in *only ten* bonds is almost exactly the maximum possible value. Of course, one is more interested in the shear forces

$(dT)_1$, $(\delta T)_2$, etc. When, with the help of Fig. 6, the expressions are developed and the foregoing value of $1-\eta=1/2661$ is substituted, one finds—

$$\begin{aligned}
(\delta T)_1 &= 0{\cdot}562\ c'e\\
(\delta T)_2 &= 0{\cdot}246\ c'e\\
(\delta T)_3 &= 0{\cdot}108\ c'e\\
(\delta T)_4 &= 0{\cdot}047\ c'e\\
(\delta T)_5 &= 0{\cdot}021\ c'e\\
(\delta T)_6 &= 0{\cdot}0090\ c'e\\
(\delta T)_7 &= 0{\cdot}0038\ c'e\\
(\delta T)_8 &= 0{\cdot}0017\ c'e\\
(\delta T)_9 &= 0{\cdot}0007\ c'e\\
(\delta T)_{10} &= 0{\cdot}0003\ c'e
\end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

It is of very great interest to note that, with the *most flexible* arrangement of crossing fibres (Fig. 5), more than half of the ultimate tension developed in

NUMERICAL FACTORS OF j AND ITS POWERS IN EXPRESSIONS FOR:

T_{n-q}

q									
0									
1	1								
2	3	1							
3	6	5	1						
4	(10)	15	(7)	1					
5	15	35	(28)	9	1				
6	21	70	84	45	11	1			
7	28	126	210	165	66	13	1		
8	36	210	462	495	286	91	15	1	
9	45	330	924	1287	1001	455	120	17	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

$(\delta T)_{n-q}$

										q
1										0
2	1									1
3	4	1								2
4	(10)	6	1							3
5	(20)	(21)	8	1						4
6	35	56	36	10	1					5
7	56	126	120	55	12	1				6
8	84	252	330	220	78	14	1			7
9	120	462	792	715	364	105	16	1		8
10	165	792	1716	2002	1365	560	136	18	1	9
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Fig. 6—Scheme for developing equations for T_p and $(\delta T)_p$, in which the numbering is based on $q=n-p$; counting of q begins at the fibre centre, with $q=0$ for the segment nearest the centre

The scheme is illustrated in equations (40) and (41) and, in the expression for T_{n-q}, the exponents for j range from 1 to q; in that for $(\delta T)_{n-q}$, the exponents range from 1 to $(q+1)$

the fibre occurs at the first bond (counting from either end of the fibre); at the second bond, the increment (the shear force) is 24·6 per cent of the ultimate tension—significant, perhaps, but small; beyond several bonds from either end, the shear forces are entirely negligible. In short, although the tension in the fibre is maximum at the fibre centre (and extremely close to the maximum possible value for a given orientation), the shear forces on central bonds are quite negligible. Thus, the expectation is that tension failures of bonds should occur first at the fibre ends. When an end bond fails, the next bond becomes number one and is suddenly exposed to a large fraction of the central tensile force. Depending on the statistical fluctuations of the bond strengths along the fibre, the bonds may fail in rapid-fire series or the ruptures may cease after only several failures; if the fibre curves away to an orientation of lesser strain, the bond ruptures may cease on the curve. If the bonds are of uniform strength and rupture along a straight fibre until, for example, only eight bonds of the fibre remain ($n=4$), the most remote bond from the centre would still be subjected to more than half of the maximum possible tension and the central tension in the fibre would be 95 per cent of the latter value.

If the resistance to flexure of the cross-fibres increases, because of orientation or shortening of the span, the shear force on the end bonds should increase. If orientations of the crossing fibres away from the perpendicularity shown in Fig. 5 had the effect of doubling j, for example, the shear forces at the end bonds would increase from 56 per cent to 68 per cent of the central fibre tension (which now would be within one part in 66 000 of the maximum possible value); if, because of increased extent of bonding in the sheet, s were reduced to the point that j had been increased tenfold, the shear forces at the ends of the fibre would be at the 90 per cent level and it is obvious that the second bonds would be subjected to shear forces less than 10 per cent of the central fibre tension.

The shear stress distribution in an end bond resulting from the combination of shear force arising in tension and the stress that is due to anisotropic fibre shrinkage might be of the form shown in Fig. 7. The former component corresponding to a sheet strain $e=0{\cdot}01$ (in the direction of the fibre terminating just beyond the bond), based on the numerical values employed earlier and assuming that $(\delta T)_1=0{\cdot}7c'e$, is found to be $5{\cdot}6\times10^8$ dyn/cm^2 (about 8 000 lb/in^2). This is somewhat less than the maximum shear stress arising in torque alone in bonds between fibres crossing at 45° and 135° (sheet strain of 0·01 at $\theta=90°$). Earlier, it was estimated that the stress caused by anisotropic shrinkage is of the order of 4×10^8 dyn/cm^2; the maximum of the combined stresses would then be about $9{\cdot}6\times10^8$ dyn/cm^2 (14 000 lb/in^2). If allowance is made for the fact that fibres taper somewhat

near their ends, one should increase this figure and it may be concluded that torque and tension failures of fibre-to-fibre bonds should be of comparable occurrence (but not necessarily of comparable importance).

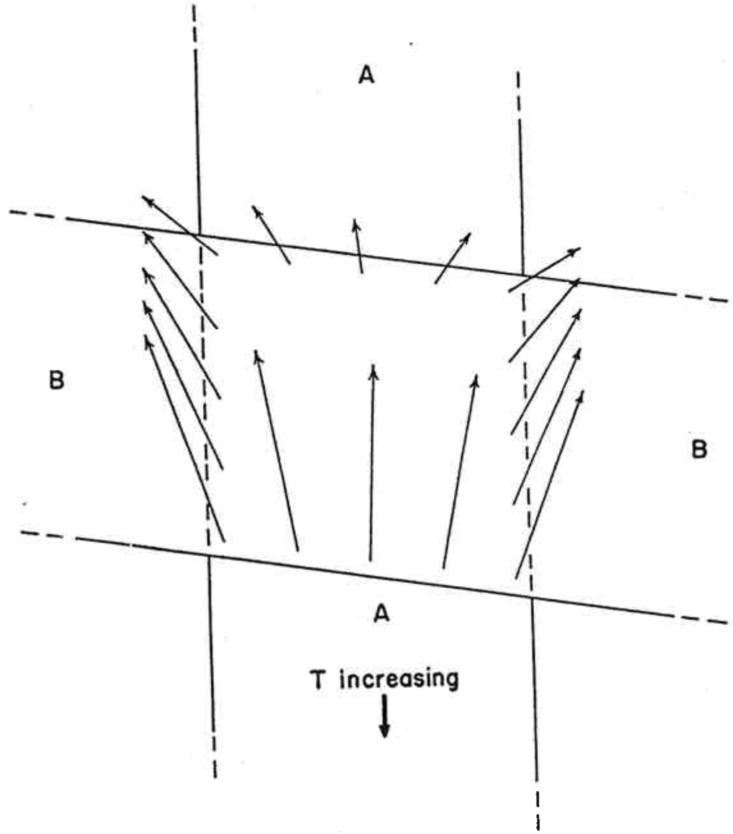

Fig. 7—Shearing forces in a bond, shown for only one fibre *A*, resulting from the combined effects of fibre tension (or compression) and anisotropic shrinkage

It is expected that tension bond failures should initiate in one of the bond edges (only the peripheral forces are shown)

Significance to the plastic regime of torque and tension failures of bonds

The considerations of the foregoing sections lead to the conclusion that, as the sheet strain increases beyond the *elastic* regime, some of the bonds of fibres in two quite different groups begin to fail. These include (*a*) some of the bonds between fibres that are approximately perpendicular to each other and oriented roughly 45° to either side of the direction of straining and (*b*) some of the bonds of fibres that are roughly parallel to the direction of straining. In the former group, bonds may fail anywhere along a fibre in accordance with the conditions of equation (*34*) and statistical fluctuations in bond strength; in the latter group, the strong probability is that the end bonds will fail first.

In assessing the effects of bond failure, it is convenient to regard the strain *e* of the sheet as the independent variable. It is useful to think in terms of the external load on the sheet, through equation (*13*) [or equation (*14*) for loading in the cross direction]. The most important term in this equation, numerically, would ordinarily be the second integral. Fibres nearly parallel to the direction of straining ($\theta \simeq 90°$) contribute importantly to the external load and are, in fact, under substantial tension; a simple calculation shows that, if the bonds held, many fibres oriented within a few degrees of the direction of straining would break when the strain is 0·03; for fibres of the size employed in earlier calculations, the tension would be 21 600 dyn (22 grams). When, at some relatively low level of strain in the plastic regime (perhaps between 0·005 and 0·01), the bonds of such fibres begin to fail, it is expected that only slight increments of strain would cause the remaining bonds of those fibres to rupture in accordance with earlier discussion. When these fibres have been taken out of play, the effect is two-fold: firstly, their mass is subtracted from *W*, a multiplying factor in equation (*13*); the angular distribution function P_θ will be modified in such manner as to place more emphasis on orientations away from the direction of straining. *The decrement is to be thought of as a subtraction from the extension of the elastic line.* Thus, although the external load increases with strain, it cannot increase with slope as great as that of the elastic line. Secondly, for each of the many bonds that fail, one segment of augmented length will appear in place of a pair of adjacent segments in a neighbouring fibre that remains in action. This, too, will cause the load/strain relationship to fall away from the elastic line.

At the same time, torque failures of bonds will occur. At the lower levels of sheet strain, such failures may not be of much importance, because, firstly, early in the plastic regime, fibres oriented at angles roughly 45° away from the direction of straining should not contribute heavily to the external load [equation (*13*)] and, secondly, the effect of each bond failure is the replacement of two pairs of contacting segments with two single segments of augmented length in each fibre—in a scattered manner, in accordance with equation (*34*)—the result of which should be only a small decrement from the elastic line. Therefore, at the lower levels of strain in the plastic regime, the effect of torque failures of bonds on the external load is not expected to be as important as that of tension failures.

As the sheet straining is increased, the ranges of angles corresponding to bond failure will increase in both groups of failure (increase of angular spread from $\theta = 90°$ for tension bond failure and from 45° and 135° for torque bond failures), as discussed earlier. It seems fairly obvious that the relative importance to the external load of torque bond failure will increase with increasing strain. Many of the fibre segments oriented within small angles of the direction of straining will have been taken out of action; *W* should be thought of as the mass of fibre per unit area that is still in action and the distribution function P_θ and P_s do not include fibres that are no longer bonded to neighbouring fibres. It is not expected, however, that the spread of range corresponding to tension bond failure can proceed far before complete failure of the sheet initiates in some spot.

One must not lose sight of the importance, in the plastic regime, of the effects of intrafibre creep and stress relaxation. The relative importance of these effects to the observed stress/strain relationship will depend, of course, on the rate of increase of the strain.

The evidence that some fibres fail during tensile straining of paper is not entirely clear;[29] if they do, one can feel reasonably sure that they involve segments making small angles with the direction of straining [tension in a fibre element can be obtained at once from equations (*1*) and (*4*)—and equation (*20*), if Poisson's ratio is not known]. If a fibre fails, there will be a decrement in the load from the elastic line, although the remaining portions of the fibre may continue to contribute to the sheet stress. In connection with fibre failure, it is of considerable interest to speculate on the effect of fibre curvature (that is, general, gradual curvature in distinction to curvature within a segment). Reference has already been made to the possibility that tension failure of bonds may initiate near the end of a fibre and propagate along a curved fibre to a point on the curve where the local segments are so oriented that the bond stress is no longer sufficient for rupture. A curved fibre may be so oriented that fibre failure under tension may occur without bond failure. The reader is asked to visualise a fibre in the form of an integral sign, with the central stem oriented in the direction of straining and the ends curved towards the positive and negative *y* directions. When the sheet is strained, there should not be the sudden build-up of tension in the fibre discussed in the foregoing section; the build-up should be *gradual around the curved ends* and it seems likely that such a fibre should be stressed to the breaking point without the occurrence of bond failure.

It will be readily appreciated that a reasonably accurate mathematical theory for the plastic regime, embodying the complex of phenomena described or implied in this and earlier sections, would be extremely involved.

It should embrace, at least in its later stages of evolution, appropriate treatment of statistical fluctuations of bond and fibre strength. It should be remembered that the model fibrous structure assumed for the present theoretical treatment is one in which each fibre segment is of such configuration that it will have tension (or compression), flexural and shear stresses linearly related to the sheet strain (that is, the model corresponds to a sheet dried under mechanical restraint). In reality, paper behaves as though the segments were not all disposed to exhibit stresses linearly related to the sheet strain; the configuration may vary all the way from Steenberg's microcrêping[15] to configurational changes of a more macroscopic nature, induced by mechanical treatment of the web. These effects, which generally reduce the slopes of the stress/strain relationship in both the elastic and the plastic regimes and increase the breaking strain, are understandable in a qualitative way on the basis of theoretical considerations like those of the present work; ultimately, mathematical theories of the stress/strain relationship should take into account both the non-linear configurations of fibre segments and of the sheet itself.

Note on the theory of the zero-span tensile test

It is of some interest to review, from the point of view of the present work, the theory[29] relating the strength of individual fibres and zero-span breaking stress. In that earlier theory, the analysis leading to the tensile force in a fibre, the introduction of the probability that a fibre will cut across a plane and the integration of components of fibre forces to arrive at an expression for the external load may be thought of as a special case of the theory leading to equations (*13*) and (*14*) (or, later, equivalent relationships). Perhaps the first thing to notice about the zero-span theory is that Poisson's ratio does not appear; this is because the clamping of the sheet by the jaws of the zero-span testing device prevents lateral contraction. The zero-span theory was evolved in connection with studies requiring handsheets, for which $P_\theta=$ const. $=1/\pi$. In work with machine-made paper, it is suggested that equation (*13*) or (*14*) be employed with the appropriate P_θ introduced in the second integral. The effect of fibre flexure and shear was not invoked in the earlier theory (this is equivalent to omitting the first integral). The only justifications for such omissions from the theory are (*a*) the zero-span test actually involves a finite span (at the instant of failure the jaws separate by an easily discernible distance) and the fibre segments in the zone in which failure evidently occurs are not 'bonded' together under the high compressive stress of the jaws and (*b*) the sheets usually tested are prepared from unbeaten or lightly beaten pulp, so that the contribution to the observed breaking load of resistance of

the fibres to flexure and shear is probably negligible. In work with strong, well-bonded sheets, it would be of interest to test the importance of the first integral.

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Appendix 1

Glossary

a Axial extension of a fibre segment
A Cross-sectional area of fibre wall
b Transverse displacement of end of fibre segment
c See equations (*37*)
c' See equations (*38*)
e Sheet strain
e_x Sheet strain in direction of x
e_y Sheet strain in direction of y
E Young's modulus of fibre, axial
$\mathscr{E}_x$ Young's modulus of sheet in x direction; see equation (*22*)
$\mathscr{E}_y$ Young's modulus of sheet in y direction; see equation (*23*)
F Shearing force in fibre segment (parallel to xy plane)
F' See equation (*34*)
$\mathscr{F}$ Shearing force in sheet
G Modulus of rigidity of the fibre for shear stress along cross-section and parallel to F
$\mathscr{G}_{xy}$ Modulus of rigidity of sheet for shear strain in xy plane and shear forces parallel to y axis—see equation (*32*)
h See equation (*8*)
I Moment of inertia of fibre cross-section referred to neutral axis in z direction
$j = sc/c'$
L Fibre length (see Appendix 2)
m Number of fibre segments per unit area of sheet
n See text leading to equations (*36–41*)
p See text leading to equations (*36–41*)
P_A Distribution function for A
P_I Distribution function for I
P_s Distribution function for s
P_θ Distribution function for θ
q See text leading to equations (*36–41*)
r Radial distance in theory leading to equation (*35*); relative bonded area—see Appendix 2
R See equation (*35*)
s Fibre segmental distance or interval between bonds
$\mathscr{S}_{1x}$, etc. Symbols for integrals—see equations (*15*) and (*16*)
T Tensile force in fibre segment, axial

$(\delta T)_p$ Increment, across bond, in fibre tension; 'tension' shearing force on bond
$\mathscr{T}_x$ Tensile load in sheet in x direction—see equation (*13*)
$\mathscr{T}_y$ Tensile load in sheet in y direction—see equation (*14*)
u Fibre width (dimension parallel to xy plane)
$\mathscr{U}$ Width of strip of sheet under tension
w Fibre thickness (dimension parallel to z axis)
W Mass of fibres per unit area of sheet
x A principal axis of sheet; corresponds to machine-direction
x_p Central deflection of a fibre element—see equation (*37*)
y A principal axis of sheet; corresponds to cross-direction
z A principal axis of sheet; it is normal to xy plane
$\mathscr{Z}$ Thickness of sheet
α Angular orientation, referred to y axis, of fibre segment bonded at 'lower' end of segment under consideration
α' Similar to α, except angle referred to line at $\theta = 45°$
β Angular orientation, referred to y axis, of fibre segment bonded at 'upper' end of segment under consideration
β' Similar to β, except angle referred to line at $\theta = 45°$
γ Shear strain in fibre, referred to segment axis and parallel to the xy plane
δ Angular orientation of fibre segment referred to line at $\theta = 45°$—see Fig. 2
ϵ See equation (*34*)
ζ Bond area—see Appendix 2
η Ratio of fibre tension to maximum possible tension for given sheet strain—see text leading to equations (*40*) and (*41*)
θ Angular orientation of fibre segment, referred to y axis
ν_{xy} Poisson's ratio for sheet, with sheet stress in x direction—see equation (*20*)
ν_{yx} Poisson's ratio for sheet, with sheet stress in y direction—see equation (*21*)
ρ Density of fibre wall
τ_m Maximum shearing stress in a fibre-to-fibre bond owing to torque on bond
ϕ_1 Angular displacement of 'lower' end of fibre segment
ϕ_2 Angular displacement of 'upper' end of fibre segment

Appendix 2

Connection between bonded area and segmental length

THE following treatment is carried out for straight fibres, although it seems likely that the final result should not be appreciably modified by moderate curvature. Consider two fibres, of length L and L' and width u and u', bonded together as indicated in Fig. 8. The apparent area of the bond is—

$$\zeta = uu'/|\sin(\theta-\alpha)| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

and the probability of the bond's existence is proportional to $LL' \sin(\theta-\alpha)$. Accordingly, for a given θ, the mean area of bonds for all values of α is—

$$\langle\zeta_\theta\rangle_{\text{Av}} = \frac{LL'uu' \int_0^\pi P_\alpha\, d\alpha}{LL' \int_0^\theta P_\alpha \sin(\theta-\alpha)\, d\alpha + LL' \int_\theta^\pi P_\alpha \sin(\alpha-\theta)\, d\alpha}$$

$$= \frac{uu'}{2\sin\theta \int_0^\theta P_\alpha \cos\alpha\, d\alpha - 2\cos\theta \int_0^\theta P_\alpha \sin\alpha\, d\alpha + \cos\theta \int_0^\pi P_\alpha \sin\alpha\, d\alpha} \quad (44)$$

In one of the steps leading to equation (*44*), an integral vanishes on the supposition that the angular distribution function is symmetrical with respect to the principal axes. On averaging all values of θ, one obtains—

$$\langle\zeta\rangle_{\text{Av}} = \int_0^\pi \langle\zeta_\theta\rangle_{\text{Av}} P_\theta\, d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

where, of course, the first factor in the integrand is given by equation (*44*).

In researches dealing with handsheets, the angular distribution function in both equations (*44*) and (*45*) is $1/\pi$ and it is easily seen that—

Isotropic: $$\langle\zeta_\theta\rangle_{\text{Av}} = \langle\zeta\rangle_{\text{Av}} = (\pi/2)uu' \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

If the fraction of the *external* surface area of the fibres involved in bonding is r, the number of bonds per unit length of fibre is evidently $2ur/\langle\zeta\rangle_{\text{Av}}$ and the mean segmental length would then be the reciprocal of this—

$$\langle s\rangle_{\text{Av}} = \langle\zeta\rangle_{\text{Av}}/2ur \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

When one deals with handsheets, substitution is made from equation (*46*) with $u=u'$ to obtain—

Isotropic: $$\langle s\rangle_{\text{Av}} = \pi u/4r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

The reader will note that equations (*47*) and (*48*) are approximations (whose accuracy improves with diminishing r), because no allowance is made for the overlapping of bonds [see, for example, the Page and Tydeman type II bond[24]].

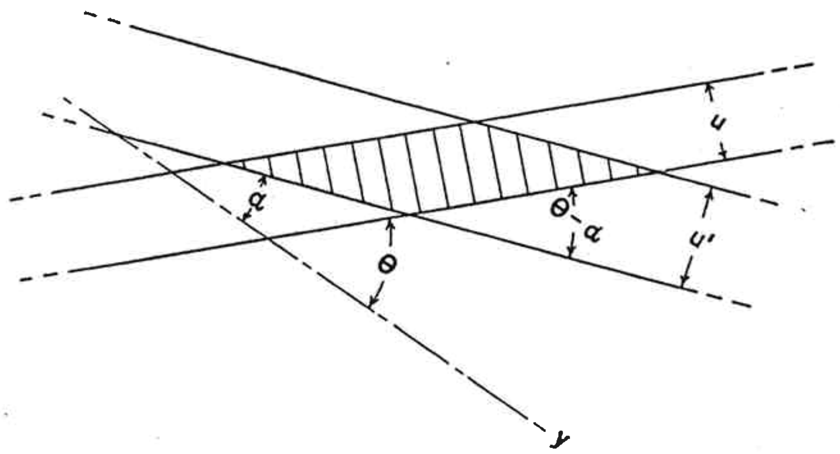

Fig. 8—Idealised fibre-to-fibre bond area appropriate to the development of equation (*45*)

Addendum 1

A paper by Litt [Litt, Morton, 'Macroscopic properties and microscopic structure in paper'; *J. Colloid Sci.*, 1961, **16** (3), 297–310], which presents a mathematical theory of the mechanical properties of paper, has appeared too late for inclusion in the author's present discussion of past work in the field. Litt's basic assumption to the effect that all fibre segments are subjected to the same force (regardless of orientation) is unfortunate. This leads, for example, to the conclusion that Poisson's ratio (xy plane) should be negative in very dense papers, which is demonstrably not the case.

Addendum 2

Dr. Charles W. Carroll has made the excellent suggestion that the products of the distribution functions be replaced with joint distribution functions. Thus, to be most general, the product $P_\theta P_A P_I P_s$ should be replaced with the function $P(\theta, A, I, s)$ because of the statistical interdependence of the variables. His contribution follows.